# MATHEMATICS IN SECONDARY MODERN SCHOOLS

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*A Report prepared for the*
*Mathematical Association*
*for consideration by all concerned in the*
*teaching of Mathematics in*
*Secondary Schools*

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A SUB-COMMITTEE of the Teaching Committee of the Mathematical Association has been considering Mathematics in Secondary Schools other than Grammar and Technical Schools since 1946. This sub-committee has been reconstituted at four-yearly intervals and few of its members have been available for the whole period. An Interim Report was published in 1949. By this time it had become clear that the sub-committee's task would occupy several years more; the short Interim Report indicated the broad principles on which discussion was proceeding and at the same time attracted valuable criticism from mathematics teachers. Further progress was made by the sub-committee serving from 1950 to 1954, but in the later part of this period the work was deliberately slowed to allow for the passage of the Report on the Teaching of Mathematics in Primary Schools, which it was felt should precede this one since the Primary Report worked out in detail for the earlier years of a child's mathematical education the fundamental principles which are equally applicable at the secondary school stage. The present Report should thus be read in conjunction with and as a continuation of the Primary Report. During the period 1954–1958 the draft scheme of the Report and sample chapters were circulated to the Association's Branches and to some individual members for discussion and criticism; the sub-committee thus had at its disposal constructive suggestions from an unusually wide range of teachers, whose valuable help is here acknowledged. The sub-committee is indebted to other members of the Teaching Committee for many suggestions and to members who served during the years 1946–1954, particularly the former Chairmen of the Teaching Committee, Mr. J. T. Combridge and Mr W. J. Langford, and also Mr. K. R. Imeson, Mr. (now Professor) W. W. Sawyer, Mr. C. W. Tregenza, Professor F. W. Wagner, Miss M. E. Bowman, Mr. M. W. Brown, Mr. C. T. Lear Caton (died 1947), Mr. J. W. Cowley, Mr. J. C. Skinner, and Mr. I. R. Vesselo.

MEMBERS OF THE MODERN SCHOOL SUB-COMMITTEE 1954–1958:

(dates in brackets show date of joining this or previous sub-committee)

MISS L. D. ADAMS (1947) (*Chairman since* 1954)
MISS H. M. COOK (1946)
DR. P. C. DAVEY (1954)
MR. B. J. F. DORRINGTON (1950) (*Secretary of the Teaching Committee*)
MISS Y. B. GIUSEPPI (1950)
MR. R. E. GREEN (1954)
MR. E. J. JAMES (1954) (*Secretary since* 1956)
MR. A. P. PENFOLD (1954)
MR. A. W. RILEY (1946) (*Secretary* 1946–1956, *now Chairman of the Teaching Committee*)
MISS K. M. SOWDEN (1950)
MR. T. R. THEAKSTON (1956)
MRS. E. M. WILLIAMS (1946) (*Chairman* 1946–1954)

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## *Note on the numbering of paragraphs*

Decimal notation has been used to indicate the main sections of a chapter and the sub-divisions of each section. For example in Chapter 4 all paragraphs are numbered with unit 4; the first decimal figure indicates a major topic as shown in the Table of Contents; further decimal figures are used for sub-divisions of the topic. In Part II many of the major topics occupy long sections; where this is so, the topic begins with a list of its chief sub-topics. Thus 6·4 is *Graphs*, as shown in the Table of Contents and 6·43 is *Distance-time Graphs* as shown at the head of 6·4.

## *Note on references to other reports of the Association*

Several of the previous publications of the Association cover topics which are relevant to Mathematics in Secondary Modern Schools. It is convenient to refer briefly to these when they treat a topic in greater detail than we do. The following are referred to, some many times:

*The Teaching of Mathematics in Primary Schools*, 1955
*The Teaching of Mathematics in Technical Colleges*, 1954
*The Teaching of Trigonometry in Schools*, 1950
*A second report on The Teaching of Geometry in Schools*, 1938
*The Teaching of Algebra in Schools*, 1934
*The Teaching of Arithmetic in Schools*, 1932
*The Teaching of Mathematics in Preparatory Schools*, 1924
*The Teaching of Geometry in Schools*, 1923

All the above are, like this publication, 'Reports prepared for the Mathematical Association' by committees of the Association's members. In referring to them full titles are somewhat abbreviated, for example to 'The Association's *Mathematics in Primary Schools*'.

# PART I

## CHAPTER 1

## THE AIMS AND SCOPE OF THE REPORT

### 1·0. Terms of Reference.

The mathematical ideas and teaching methods here discussed are those which we consider suitable for the great bulk of the secondary school population between 11 and 15 years of age, who have shown no early signs of readiness for Mathematics as an abstract study, or at least have not achieved the attainments in arithmetic traditionally associated with the ablest of their age group.

The term *modern school pupil* is used to denote such a boy or girl, though we recognise that the wide variety of school organisation from area to area means that the pupils we have in mind are to be found in all types of secondary school.

### 1·1. The Wide Variety of Schools and Pupils.

The variety of circumstances in the schools and the differences amongst the boys and girls themselves make it impossible to give any syllabus of work suited to modern school pupils in general. We therefore suggest a wide variety of possible approaches to work in Mathematics, and in Part II which deals with the Mathematics itself we include much more than could be covered by any class in a four year course. In chapter 5 we discuss problems of organisation including syllabus making.

We do not make a special feature of work for pupils whose ability is greatly below average, but we do urge that their course should be made a stimulating one and that teachers should adapt what we say about the approach to mathematics teaching to the slower pupil's pace and range. Chapter 4 deals with remedial treatment at all levels for pupils who have serious gaps in their understanding or skill.

**1·11.** Some of the extremes of difference in school organisation and circumstances are worth noting in support of our decision to make only general recommendations about syllabuses. The percentage of abler pupils educated in schools, or sections of schools, organised to deal with courses up to 18 or 19 years of age varies from about 10 to 40 per cent of an age group. Even this is not the whole picture because of the possible variations from place to place in the distribution of ability. There is therefore some overlap between what may be called a Modern School in one district and a Grammar School or grammar school stream in another. Where there is a large percentage in the grammar school category, it is likely that some of the pupils so classified would fall within the scope of our recommendations at least in the early years of their course.

Schools also differ greatly in size. Small schools may adopt particular ways of working in order to deal with differences of interest and ability within classes. Large schools have more scope in classification but may find difficulty in ensuring continuity of teaching and co-operation among a large staff.

Schools also differ in their environment and in the backgrounds and prospects of their pupils. Mathematics may appear to be a subject independent of environmental differences, but **we hold that it is essential that modern school pupils should see Mathematics as a subject that touches their lives and is worth their attention.**

### 1·2. The Place of Mathematics in the Curriculum of the Modern School.

In the Association's *Mathematics in Primary Schools* five reasons were given for including Mathematics in the education of all children. Briefly these were:

(1) Mathematics is the language of orderliness and ordered thinking.
(2) Mathematics is the tool and language of science.
(3) Mathematics is an inheritance of the race. We live in a community using the language of numbers, measurement and shapes in every day talk.
(4) Mathematics gives pleasure. Children find pleasure in mathematical thinking and achievement.
(5) Mathematics develops patterns of thinking which are fundamental patterns of all thinking.

We now develop and add to these reasons for including Mathematics in the curriculum with the education of young people in Secondary Schools in mind. Among modern school pupils as defined above (1·0) there will be relatively few who develop in later years to the stage of taking any university course or of entering professional employment. There will however be a substantial number of pupils who continue part-time education after leaving school and amongst these, especially perhaps among the boys, there will be those for whom particular parts of Mathematics will become an important tool in their chosen vocations. This vocational aspect of mathematical preparation, relevant to the work of older scholars, is discussed briefly in 3·5.

On the whole the modern school pupils likely to continue their mathematical education after leaving school will be drawn from the abler section amongst them, say the top third. Their most important need vocationally is a sound foundation of understanding of whatever has been attempted at school and confident skill in such techniques as they have learned. No Modern School can hope to do more than give its older scholars a little insight into the great variety of special forms of Mathematics used in different trades and professions, for example in the building trade or nursing. Still less can it hope to find in existing external examinations in Mathematics anything suited to any large number of its pupils.

It is then as a part of the equipment of the ordinary citizen that Mathematics must first be considered in the Modern School. Its use as a specialised tool may be needed by some later in life, but readiness for many mathematical techniques comes with an intellectual maturity only reached by most modern school pupils some time after entry, if at all.

**1·21.** For the ordinary citizen Mathematics is first and foremost a living language. Indeed on utilitarian grounds there is little to be said for developing mathematical techniques beyond what is expected of abler pupils at 11 years of age as this covers most of the needs of everyday affairs. On a citizen taking an intelligent interest in the world around him however the demand for an understanding of mathematical expression is considerable and increasing. There are mathematical ideas prevalent in the world today which have so far found little or no place in school text-books.

Traditional courses in English schools were planned for a nation of shopkeepers and applications to commerce and trade formed a very large part of them. Such a calculation as 'Find the cost of 8 cwt. 3 qr. 17 lb. at £3 17s. 6d. per cwt.', taken from an old text book, would now be accomplished, if required at all, by a machine in all but the smallest offices. Instead of performing this calculation the operator must be able to decimalise the quantities concerned before 'feeding' the machine. Curiosity may be aroused in pupils as to how these machines work instead of making them labour over calculations no longer used. In smaller concerns such calculations as were required would be performed with the aid of a ready reckoner which uses the method of calculation by parts. Nothing is required of the user beyond the ability to add and subtract and perhaps to recognise the simplest fractions. Even counting and measuring is now done by self-recording machines, and the use of slide rules by engineers is well known.

Nevertheless the teacher must bear in mind that machines have to be designed and fed, and that they cannot record or calculate anything unless the idea of what is wanted has first been in some human mind. It must not be lightly assumed that the ability to use a machine to carry out either one's own purposes or those of someone else can be all that the mathematics teaching of the Modern School should aim at, even if some pupils are allowed to acquire this ability before they have gone far in other work and much earlier than has been usual. Moreover the Mathematics of the future may be different again within the life-time of a pupil.

**We ask that the traditional mathematics syllabus be regarded very critically and the purpose for which any of it is included be reviewed.**

After consideration of the reasons (1·2) for including Mathematics in the curriculum of all schools we give below some mathematical themes which develop these ideas, and which may be given some precision in the mathematical classroom.

**1·22. Development of Mathematics as the language of Science.** Mathematics as the language of Science has developed differently from Mathematics as the language of Commerce. On the numerical side both are based on the ability to count in tens, but the numbers required in commerce are of very limited range, whereas the scientist's mind must adventure into the realms of the very large and the very small. In popular texts on astronomy or atomic structure there are references to very large and very small numbers, and the modern citizen should be trained to understand the notations in which they are expressed (6·247). Such comprehension is more important to him than, say, the ability to calculate with decimals, but comprehension is a slow growth and has to be stimulated by experience and significant usage. The story of the development of notation may help

**1·23.** The scientist has also devised ways of measuring quantities for which the tables of weights and measures as traditionally learnt are by themselves quite inadequate, and has also systematised the measurement of length, weight and capacity in the metric system. Some history and theory of measurement will give new meaning to units already familiar by name, as well as making clear the reasons for learning about new ones. Ideas of how to measure such intangible quantities as power, wind strength, heating capacity, electric current and resistance require units of new types; wave lengths, pressure, speed, petrol consumption, density, use old ones in new relationships. There are also measures which are expressed as ratios independent of units; these include chance, specific gravity, cost of living index, representative fraction of a map, coefficient of friction or of restitution. Indirect and alternative forms of measurement are of interest, for example gas consumed is recorded on a meter and may be assessed either in terms of cubic feet recorded or of therms available from burning them (6·81 etc.).

Techniques of direct measuring need not be confined to the usual foot-rule, protractor, scales and dynamometer of the classroom but can include the handling of many instruments: for length, a vernier scale and a diagonal scale; for angles a clinometer and a theodolite; for weight, a steelyard; as well as others such as a micrometer, a sine bar, a pantograph, a spherometer etc. All this may lead to fruitful mathematical exploration into their principles.

As skill in measuring is developed interest may be aroused in the relative accuracy of measurements. The pupils themselves may with difficulty estimate to a hundredth of an inch, but they can hear of machines which can work with precision to one millionth of an inch. In contrast, astronomical distances are so great that they can be conveniently given only in millions of miles, or even in still larger units, viz. light-years, yet calculations forecasting the positions of far distant stars relative to the earth are made to fractions of a second of arc (6·82).

**1·24.** Again, a study of change might compare rise and fall, increase and decrease, expansion and contraction, growth and shrinkage, appreciation and depreciation, progress and regression, as well as profit and loss. Opportunities can be found in commercial practice and in scientific study, and investigation can range far and wide over the modern scene. Such topics as simple and compound interest, biological growth, mortgages and hire purchase, sinking funds, saving certificates, size and intensity of an image on a screen at varying distances, radio active decay, salary scales, cost of living index, show change in great variety. Mathematically it is important to become accustomed to different ways of examining and describing changes, to growth factors as well as differences, particularly to a constant factor, and to the significance of graphical representation of changing quantities.

**1·25.** Citizens of to-day are also interested in probability and chance, and here too Mathematics is relevant. The probability of winning in a lottery or a raffle may be compared with the principles on which premiums to cover risks of fire, or transport accident, are calculated. If such study is carried far it will of necessity be linked up with some gathering of statistics, the examination of statistical tables, and the finding of averages, medians, and other statistical measures (7·5). Information made available to the public is often expressed in the form of diagrams and graphs, and understanding the language of this form of representation is of great value to any newspaper reader (6·4).

**1·26.** Most of the foregoing examples involve numerical statement or measurement in some form, but a command of spatial ideas is equally important in coping with developments in modern life. Direction, the fixing of position, the geometry of the sphere are all required and at the same time more readily understood because of the extent and conditions of modern travel; the sphere is compassed in imagination and the mind is carried out into orbits in space. Shape is not only of static importance as in architecture but trajectories, ellipses, wave (or sine) curves have all come into common speech and thought. A fresh practical approach to geometry occupies a long section in our report (6·5).

**1·27.** Another mathematical conception which should be studied is that of a vector quantity, which requires both a magnitude and a direction to define it, and is not subject in the ordinary way to the four rules of arithmetic. Displacements, velocities, accelerations, and forces can be combined, or in a new way 'added', or their effects can be considered in two parts by resolving along two directions at right angles. If a topic such as navigation is being studied, even simply, familiarity with vectors and facility in adding them by the triangle (or parallelogram) of vectors will be acquired (7·25).

### 1·3. The Urgency of Important Changes.

We have given in some detail mathematical ideas and techniques related to modern developments because it is at the present time a matter of urgency to make citizens aware of the relevance of Mathematics to their everyday lives. We are convinced that many pupils who do not make much progress in Mathematics at the primary stage, or even in the first year or two at a secondary school, may, as they become more mature and feel the urge to solve some particular problem, find that some form of Mathematics meets their need, and from this beginning they will develop their mathematical language into a constructive tool, and perhaps even into a mode of thought.

### 1·4. The Plan of the Report.

Our report is divided into two parts, but the chapters are numbered consecutively and it is hoped that it will be regarded as a unity: the stress in the two parts is indicated by their titles 'Purpose, Approach and Organisation' and 'Mathematical Content', but there are constant cross references, and it has been impossible to avoid a certain amount of repetition. An index of the mathematical terms and topics of Part II is provided, as it is hoped that this part of the report in particular will be useful to teachers for occasional reference. The index also shows where mathematical topics occur in illustrations of the more general questions discussed in Part I.

In chapter 3 and also in chapter 7 we try to give a representative sample of practical topics through which Mathematics may be learned, and for the study of which some Mathematics is essential. Those in chapter 3 are chosen for their intrinsic interest and importance and we show how they may be used as an approach to or an application of Mathematics: in this category we include the economics of everyday life because of its social importance, and man's use of the air as an example of a topic of contemporary interest. In chapter 7 the topics dealt with have practical techniques of their own, but they are predominantly mathematical, so that their study can form a basis for a progressive course in various aspects of Mathematics. Details of two topics, viz. the drawing of certain curves and a technique of statistics, which are less familiar to teachers because not usually included in courses for pupils under 16 years of age, are set out in two appendices.

Chapter 2 has something in common with chapters 3 and 7 in stressing topical interest but it is designed to show how teachers responsible for entrants' classes may combine freshness of approach with their need to find out what they can about their pupils' mathematical background.

Pupils entering the Modern School are generally quite unready for the kind of mathematical syllabus associated with grammar school courses. Some may attain readiness for an intellectual approach later in school life, but many will never do so. Chapters 2, 3 and 7 and chapter 4 on remedial work contain many suggestions intended to fill this gap between the work of the Primary School and the courses of the Secondary Grammar School.

### 1·5. Continuity and Development at the Secondary Stage.

The Association's *Mathematics in Primary Schools* emphasises that the only sound basis for mathematical techniques is understanding and that this understanding is developed from the children's active response to experience, that is, from experiment and discussion rather than precept. This ideal is still far from realisation so far as a majority of children leaving the Primary Schools is concerned because of tradition and circumstances which involve teaching to all what may be suited only to a few.

The principle that the kind of understanding that arises from experience should precede the development of techniques still holds good in the Secondary School, though, as pupils mature mathematically, tangible and direct experience plays a smaller part and in the later years of school life more and more of the work becomes a development from mathematical experience itself.

**The Secondary School is moreover increasingly influenced by the approaching end of full-time schooling and by the demands of adult life as well as those of individual growth and need. The pupil's own sense of values is more and more based on his estimate of what will serve him in good stead in the future rather than bring him immediate satisfaction and approval.**

**The rapid and continuing changes in world conditions and interests, and the growing importance of Mathematics as a language for the description of new experiences, lead us to suggest that there must be new emphasis in choosing what is suitable material to present to pupils at all stages, and that adaptability and understanding, in whatever range of work is undertaken, are qualities to be prized more highly than ever before.**

## CHAPTER 2

## ENTERING THE MODERN SCHOOL

### 2·0. The Importance of Freshness of Approach.

We regard it as exceedingly important that a pupil entering a secondary school should have an opportunity to make a new approach and to get as much thrill as possible out of this fresh and exciting experience.

Arithmetic has played so large a part in selection for the new school that the teacher of Mathematics has particular difficulty in keeping a sense of novelty in the forefront. Different teachers will solve this problem in different ways. Some will depend on stimulating fresh interest in familiar material by approaching it in a new way, while others will make use of something new in Mathematics which at the same time reveals the pupils' grasp of familiar work. Some simple statistics or surveying for example might well serve both purposes by introducing new mathematical ideas and at the same time stimulating interest in familiar computation or drawing.

In this chapter we discuss ways of making a fresh approach to mathematical material commonly included in syllabuses, but many later suggestions of a more varied kind, particularly in chapters 3 and 7 would lend themselves to a quite elementary treatment suitable for entrants.

**2·01.** The Head of a Secondary School classifying a group of entrants or the teacher of an entrants' class is of necessity concerned with assessing what has gone before. Diagnosis must go deeper than a mere assessment of whether on the day of entry pupils can recall this or that item of information or perform accurately certain procedures of arithmetic. Indeed direct testing of such matters at the outset has some grave disadvantages especially for pupils who are already suffering from any sense of failure. Enough information about the mathematical teaching and the pupils' performance in the Primary School should be available to guide the first placing into classes of all but an occasional entrant. In using the records of individuals from the contributory schools, Heads of schools can usually interpret the reports from one school in relation to those from others.

Other general information about the Primary Schools will be useful not only in classification but also in deciding how to deal with entrants, for example one school may be known to encourage initiative and experiment whereas another may concentrate on drilling facts and procedures to such an extent that its pupils show little independence of thought though they work accurately when told what to do.

As well as assessing the stage reached by a pupil in Mathematics the teacher should try to discern the child's attitude towards the subject and appreciate such qualities as confidence in attacking problems, confidence and accuracy in carrying out well-known processes, and initiative in facing new situations.

Children whose work is hampered by uncertainty, over-dependence, discouragement or confusion will benefit greatly from the freshness of approach referred to in our opening paragraph and dealt with more fully below.

Although the need for remedial work may become apparent very early we do not recommend either that tests should be set at this stage or that remedial work should be anything but incidental to new and interesting work in the opening period of the course. After some weeks of such work a great deal more should be known about individuals and about the class. In making an estimate of pupils' abilities and needs, work can be planned so that much of the revision and remedial work that is necessary, as suggested in chapter 4, appears a normal part of the work of the class or of groups within it.

We give in some detail in the next section an example of a well-tried introductory activity, and alternative ways of beginning are suggested in 2·2 and 2·3.

### 2·1. An Example of an Activity for Entrants, 'Myself and My New School.'

For the purpose of this example, it is supposed that the school concerned is a 3-stream Modern School in a country town drawing boys and girls from three sizeable town and semi-rural schools and from several small country schools. The class particularly considered is supposed to be the 'B' class, whose pupils have been assigned to it from the Primary Schools' orders of merit and school records; unless specially recommended the pupils from the very small schools are supposed to have been placed in the B and C classes for promotion later if they show promise. Some classes of pupils might pass on in a week or two to the new investigations which the topic suggests; others might continue regular work connected with it for a term or more.

**2·11. Organisation of the work.** Classwork is occasional only. The pupils work individually, in pairs or in groups of six to eight members according to the nature of the job. The group is the main unit of organisation. Groups are carefully chosen to include pupils of differing backgrounds, e.g. at least one from a country school and one from a particular town school known to give its children good experience with measuring instruments. So far as can be determined from records groups are chosen to have a range of ability with at least one pupil in each likely to show some leadership.

The problems for the teacher in organising group work are discussed in 5·6. If preparation has been adequate, and the teacher's manner is confident and arouses interest, difficulties with discipline are unlikely.

Pupils are told that they will compile folios of information about

(*a*) individuals, 'myself' e.g. age, measurements, home, etc.,
(*b*) the group.
(*c*) the class,
(*d*) the school.

In arithmetic lessons this information will be recorded numerically and graphically, but other information may be collected in other lessons.

A background activity is also included to keep pupils busy while waiting for advice or materials. Each child has an exercise book entitled *What I Know* in which to record information or work examples to show what he has already learned in arithmetic. Teacher's cards and a textbook are available to suggest questions and examples. Answers can be looked up at agreed intervals and inspected by the teacher, who can at any time switch a pupil from free to directed work.

**2·12. The collection of individual statistics.** Individual statistics, if collected on separate sheets or cards can be dealt with in any order. Some class discussion of measuring instruments to be used in finding personal measurements is probably worth while, after which instruments can be allotted in turn to groups within which the pupils can work in pairs. The teacher might well take a period to talk with each group in turn while the other groups either make forecasts of what they can record under the headings (*a*), (*b*), (*c*), (*d*), of the previous paragraph or make entries in their *What I Know* books. Class discussion should decide which individual measures should be dated because they are variable. Useful comparisons can be drawn between the variations to be expected in weight and in height.

The investigations suggested below are of varied difficulty and some will require closer supervision by the teacher than others. A group which has just mastered one type of record can usefully help another group just starting it.

**2·121. Weight.** The finding of individual weights is a part of the routine of medical inspection. Dated records may be available from medical cards but each group working in turn with the teacher should find the weights of at least two or three individuals on the medical balance. Discussion of balances and units of weight might be supported by other actual experience of weighing. (If the medical records are made in kilograms there is opportunity for a new experience of measures of weight.)

Records can be made in stones, pounds and fractions of a pound and in pounds only. Pupils can be encouraged to write what else they know about weight in their *What I Know* books. The teacher has the opportunity here to discover the pupils' powers of dealing with the numbers 14 and 16.

The validity of giving a child's weight, which alters from hour to hour, to perhaps a fraction of an ounce, should be discussed.

**2·122. Age.** A card with blanks to be filled in can be given to each pupil, but written or oral advice will be needed. A card completed in September, 1958, would appear thus :

| Name | Date of Birth | | | Age (on same day of month) in September 1958 | |
|---|---|---|---|---|---|
| | Day | Month | Year | | |
| John Smith | 12 | 7 | 46 | On Sept. 12th | 12 yr. 2 mth. $=12\frac{2}{12}$ yr. |
| Ann Brown | 31 | 10 | 46 | On Sept. 30th (31st) | 11 yr. 11 mth. $=11\frac{11}{12}$ yr. |

The months in the ages will probably be reckoned by the pupil by counting on from the previous birthday or back from the next. The fractional statement may be understood and used by only a few.

When the individual cards are complete, and possibly before some of them are checked by the teacher, the class can enjoy checking months and years by the simple device of seeing that the years and months should add up to 58 and 9 or 57 and 21. In the examples shown the checks are:

| | | | |
|---|---|---|---|
| months | $7+2=9$ | and | $10+11=21$ |
| years | $46+12=58$ | | $46+11=57$ |

There is excellent opportunity here for oral work though pupils' explanations are unlikely to be complete and well ordered. The teacher will note each pupil's knowledge of the calendar, his use of fractions and any initiative shown in planning.

**2·123. Length.** The pupils might choose which of their body measures they will all record, but height should be included and at least one measure of girth. Some of the body measures which in adults are associated with named units might be chosen; for example span, hand, palm and foot. The rigid measuring instrument for straight lengths, e.g. the tape fixed vertically on the wall and the technique of using it to measure height, and the flexible measuring tape for measuring the waist should be contrasted.

Results should be (*a*) stated freely
(*b*) stated to nearest ½ inch
(*c*) stated to nearest ¼ inch
(*d*) stated as accurately as the tape allows using feet, inches and parts of an inch.

Some pupils may omit some of these details and make a variety of statements in only a few cases, but the teacher will look ahead to see that he has what is needed for the group discussions of 2·131. The teacher has many chances of noting the use of fractions and of measures, and the measurement of height involves various geometrical ideas such as perpendicularity, parallelism and vertical and horizontal lines.

**2·124. Time.** 'How I spend my day' can be examined by first recording the times at which the child gets up, leaves home, arrives at school, breaks off for dinner, and so on and so calculating the lengths of time occupied in various ways. These will first be stated in hours and minutes but for diagrammatic representation of the main divisions of a 24-hour day, the times should be stated to the nearest quarter of an hour. Even so there will be difficulties of scale unless a long time line of 12 or 24 inches is used. The diagram might be of the kind shown in Diagram 1.

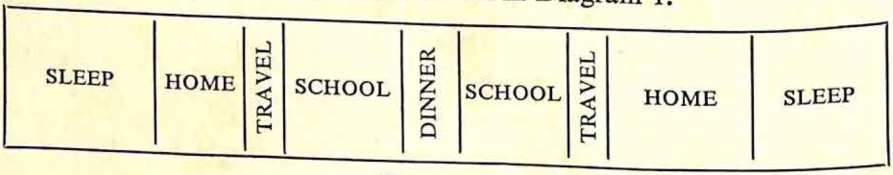


DIAGRAM 1.

The teacher will note the pupil's ability to tell the time and should follow this up with alternative ways of stating a time such as 6.40 p.m. or 18.40 instead of the more familiar *twenty to seven*. In reckoning intervals of time there is opportunity for work on fractions of an hour and the factors of 60. The simple idea of ratio may appear when the child compares the intervals of time spent in different ways. It might be well to start with simpler diagrams than that shown and to proceed to those showing more detail or covering longer periods of time. Discussion will be needed between the attempts by a group to make records and the drawing of diagrams.

**2·13. Group and class statistics.** A good deal of work must first be done in collecting such individual statistics as number of children in family, number of girls and boys, the place of the child in the family as regards age and, at a later stage, the distance of home from school, the length of an average pace (2·16(*c*)), and speed of walking.

These statistics can then be used for drawing block graphs where appropriate and for work on averages. The interest and scope of the work will be greatly increased if statistics for other classes are available for comparison.

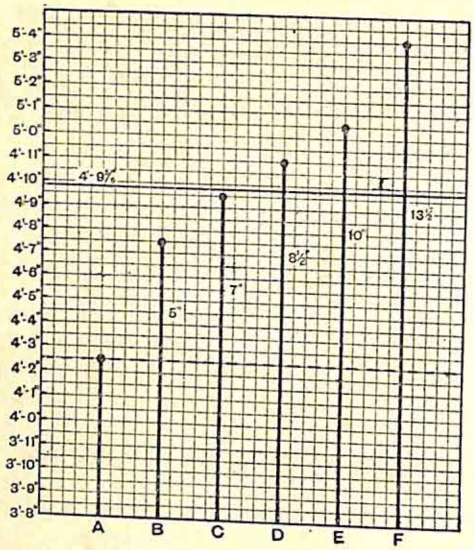
**2·131. Averages.** The treatment of the group statistics for age and for height is now suggested in more detail.

The range of age depends on the school organisation, but it probably runs from about 11 yr. 0 mth. to 11 yr. 11 mth. The distribution may be considered in relation to the central figure 11 yr. 6 mth. for working groups and for the class. How many above 11 yr. 6 mth.? How many below? How many months above? How many below? Does this suggest anything about the age of the group or class? How does this compare with other groups or classes?

From a class discussion of such points some idea of the concept of average should begin to emerge even if the name itself is not yet used. The idea can be further developed along some such lines as the following suggested work on the children's heights.

Select a group of 6 pupils which is *not* a working group, so that working groups may later do a similar exercise; then:

(*a*) Note, in any order, the heights to the nearest half-inch;
(*b*) Arrange in order of magnitude.
(*c*) Make a diagram either of the heights or of their excesses over, say, 3 ft. 6 ins. (Diagram 2).
(*d*) In the diagram draw a horizontal line through the top of the smallest upright thus showing the excess of each height above the shortest.
(*e*) Work out these excesses in inches, find their sum and add a sixth of it to the height of the smallest child, pointing out that if each were of the height so obtained the total height would be unchanged. The calculations involved here can be avoided by marking the excesses as shown on the diagram along the edge of a piece of paper, dividing into 6 by folding and placing one sixth at the top of the smallest column.
(*f*) Explain that this ' levelled out ' result is what is called the *average height* of the group.
(*g*) Check the result by finding the totals of the differences of individuals' heights above and below the average, which totals should, of course, be equal and will be approximately so in this case.
(*h*) Draw a line on the diagram at this average height.



FINDING THE AVERAGE HEIGHT OF 6 PUPILS

A—$4'\,2\frac{1}{2}''$
B—$4'\,7\frac{1}{2}''$
C—$4'\,9\frac{1}{2}''$
D—$4'\,11''$
E—$5'\,0\frac{1}{2}''$
F—$5'\,4''$

RECKONING FROM $4'\,2\frac{1}{2}''$

$0+5+7+8\frac{1}{2}+10+13\frac{1}{2}=44$

AVERAGE HEIGHT $=4'\,2\frac{1}{2}''+\frac{1}{6}\text{ of }44''$
$=4'\,2\frac{1}{2}''+7\frac{1}{3}''$
$=4'\,9\frac{5}{6}''$

DIAGRAM 2.

An alternative treatment after drawing the diagram for (*c*) above is to cut out the columns, give each child his own, then join them end to end and divide into 6 equal parts by folding. This method might be a preliminary for all pupils but might serve as the main method at this stage for slower pupils.

The groups may now get to work on finding their own group averages for height, age, weight, number in family and so on. They should compare the ease of computation when working

(i) from a 'spotted average'
(ii) from a round number
(iii) from zero.

The method which is the easiest for computation is not necessarily that which gives the clearest idea of average. Perhaps only abler pupils would complete the whole sequence, when this idea was new to them, and master it sufficiently to tackle comparative methods.

**2·14. Costs.** There are many opportunities for keeping accounts in school in connection, for example, with dinners, cookery, uniforms and national savings, and these accounts may apply to individuals or to classes. As a group exercise the pupils can find the cost, for example, of dinners for the group for 1 day, for 1 week or for a whole term. If uniforms are worn the cost for a group may be found in two ways as an example of a self-checking exercise. For a group of 6 pupils the total cost would be found of (i) 6 blazers, 6 badges, etc. and (ii) 6 sets of uniform. The teacher will note whether multiplication is sufficiently well established to be used instead of addition. Pupils will tend, in a practical situation such as this, to revert to the earlier known and simpler though longer method of addition.

Accounts may have to be kept for needlework and other craft materials, and although payments by pupils are not involved, some costing of an inventory of books and stationery may help to give the children a sense of values as well as giving opportunity for practice in stocktaking.

Measures as well as money may be involved, for example, in the cost of supplying free milk for the class for a week or a term, where the total amount may be measured in gallons.

A special activity such as a visit to the country for the benefit of the town children or a visit to a place of interest in town for those from the country would involve questions of sharing expenses.

**2·15. School statistics.** Any statistics available for the school as a whole under previous headings would be useful especially for abler pupils who had mastered ways of representing statistics graphically or of finding averages of smaller groups.

The examples given below of school investigations include some items best noted as individual records, e.g. average pace is included with the finding of longer distances.

**2·16. Longer distances.** Fresh techniques of measurement are needed in finding longer distances. Significant distances can be chosen such as the distance of the classroom from the hall, the lengths of games' pitches, etc. Class discussion is required of some of the means of measuring, but detail should be worked out by the groups. The organisation can be by groups in turn using the same method, or by groups using different methods for different distances simultaneously and then exchanging jobs. Again organisation may be eased if some are ready before others for practical work in this section.

The main methods available are:

(*a*) Yardstick and guiding string. Probably the need for a guiding string to determine a straight line will not be immediately obvious.
(*b*) Footing, that is walking the distance putting heel to toe.
(*c*) Pacing. The finding of the length of an individual's average pace (not stride) is in itself a most valuable exercise. Groups should organise their investigations themselves and submit their methods to the teacher for comment. Significantly different average paces can be used to make separate estimates of the same distance.
(*d*) Using the trundle, a wheel of known circumference, which rolls without slipping over the distance to be measured (7·111e). This method is useful for distances which are not straight, but it may be wiser to postpone it unless there is available a trundle with a circumference which is a simple exact length. The difficulties of multiplication and of possible error which otherwise arise provide useful exercises at a later stage.
(*e*) Chain or tape. The tape may be either homemade or a standard piece of apparatus. If the length to be measured is greater than the chain the problem of maintaining a straight line arises. Some distances found by other methods might be checked with the standard tape. (7·111.)
(*f*) Local maps. Maps might be used in finding some longer distances.

The units known to the pupils should be noted and their range extended if necessary to include

$$\begin{aligned} 1 \text{ chain} &= 22 \text{ yards} \\ &= 100 \text{ links} \\ 10 \text{ chains} &= \tfrac{1}{8} \text{ mile} \end{aligned}$$

The work should be continued until every pupil has had some experience of a variety of units.

**2·17. Distance, time and speed.** An interest in the time taken to walk from home to school might lead to an approximate method of expressing distances on the assumption of a standard speed of walking, e.g. 10 minutes' walk from the station. Thence interest might turn to the longer distances travelled on the school 'bus where time schedules and distances measured on the local large scale map will enable speeds to be calculated. Class discussion of the relationship between distance, time and speed may lead to individual experiments and reveal to the teacher the extent of the children's real appreciation of a distance such as $\frac{3}{4}$ mile or of a speed of 30 m.p.h. After some preliminary discussion these ideas can be developed as far as seems appropriate at this stage.

**2·18. Spatial knowledge.** Some incidental opportunities for diagnosis of spatial knowledge have already been mentioned, viz. perpendicularity in connection with measuring height, the straight line in measuring long distances and the rolling wheel. Other opportunities can be introduced for their own sake. The observation and naming of shapes, plane and solid, in and around the school building would give a good idea of the extent of the pupil's vocabulary for geometrical form and might lead to further discussion. The question 'Where do you sit in your classroom?' would probably result in incomplete and poorly expressed answers which could be improved by giving *co-ordinates* of a desk in rows and columns from two intersecting walls or by giving *distances from* these walls. The range of possibilities depends on the particular environment.

**2·181.** An activity is described in 4·811 which might well arise in discussing the school building, viz. a comparison of the sizes of a classroom and the hall, and this would lead through an investigation involving lengths and shape to the idea of area. Although this is placed in the remedial section many entrants might find it refreshingly new and the numerical part of the treatment could be used diagnostically as well as remedially.

## 2·2. An Alternative Example.

Freshness can sometimes be given to the work simply by a change in the way of working. Suppose for example that pupils have had little or no experience of using measures in practical situations and that work in the Primary Schools has been mainly by class instruction and working examples from text books. It would for such pupils be a refreshing change to work individually or in small groups and to do measuring of a practical kind on some such simple plan as that outlined below. Work of this kind is not uncommon in Primary Schools and for those pupils who have already worked in this way the effect of such an approach might be discouragingly repetitive rather than refreshing.

Work is planned in sections with no unifying theme. Sections are:

1. Weighing.
2. Measuring with ruler, tape measure, yard stick, 66 foot tape.
3. Measuring with gallon, quart, pint, ½-pint, gill.
4. Counting singly, and in groups; notation.
5. Counting of money and grouping into larger units.

For each section a series of work cards is made giving tasks or problems which might arise naturally.

A wall chart is made, divided into rectangles, showing all tasks to be completed down one side and the names of the children across the top. Each child is responsible for putting a tick in the appropriate space as he finishes a work card. He then chooses his next task from the work cards not in use. By arranging that there is only one card for each set of apparatus available, the teacher can give freedom of choice to the children without having any resulting confusion or argument between two or more children.

*Examples of tasks or problems in each section.*

1. *Weighing.*

Find the weight of this parcel and how much the postage would be (using the parcel post rates' table).

2. *Measuring: Length.*

We want to put pictures on a rail to be fixed round the classroom wall. How many feet of pin-rail must we buy and what will it cost at ...... per foot?

3. *Measuring: Capacity.*

How many ½ pint glasses of lemonade would the large enamel jug hold?

4. *Counting.*

How many exercise books are there in one of the packets in the stock-room? How many packets? How many exercise books altogether?

How many blocks (tiles, strips of wood) in the floor of the classroom?

5. *Counting: Money.*

How much money in £ s. d. is there in the box of half crowns?

### 2·3. A Beginning with Work Mathematically New.

Freshness of approach can also be achieved by teaching something which is mathematically new to the pupils. New work generally also throws light on previous knowledge.

Work of a geometrical kind is often new or almost new to many entrants. Early stages of surveying (7·1), are for example suitable. Ways of measuring longer distances, as in 2·16, are a beginning. Early stages of the study of angles and their measurement are discussed not only under surveying but also in 6·511. Again the geometrical intuitions of the eleven year-old are adequate for problems of the 'buried treasure' type in which simple loci are used. For example treasure may be buried 10 feet from the tree and 20 feet from the white stone etc. Fixing position by distances from lines is also suitable. So is some of the work with right-angled triangles which forms the basis in learning trigonometry (6·6).

One reason for teachers' reluctance to introduce geometrical topics early in the first year is that they feel that consolidation of the number teaching of the Primary School is of over-riding importance. Yet in fact a geometrical approach can provide a surprising amount of opportunity for numerical work as well as giving a lively approach to new problems.

In the first place anything which involves a scale drawing in which the scale is not of the fundamentally simple type, 1 inch to 1 foot, involves a multiplication–division relation; for example, if the scale is 1 inch to 2 feet then 30 feet is represented by $(30 \div 2)$ inches, and 3 inches represents $(3 \times 2)$ feet.

For pupils of good ability who will certainly include decimals in the secondary school course even the earliest scale drawings may be used to give valuable experience of the notation of decimal fractions; lengths can be measured in inches and tenths, and scales employed which involve fifths and tenths (6·247).

Again in finding an average pace as suggested in 2·16(c) it is usual to count a convenient number of paces, say 10, and measure the distance paced. But the information may be recorded in either of the two forms

$$10 \text{ of my paces} = x \text{ feet},$$

or, if the distance was measured before pacing,

$$y \text{ of my paces} = 50 \text{ feet}.$$

The single pace is then found by division. The form of the answer depends on the child's number competence; it may be given in feet and inches or as a fraction or decimal fraction of a foot.

In using the average pace to estimate a distance multiplication by the number of paces is involved. Since this is generally taken as a whole number no special difficulties arise though the work may be quite heavy. For example if the pace be given in fractions as $2\frac{5}{12}$ feet, 43 paces may be thought of as giving $(2 \times 43)$ feet and $(5 \times 43)$ twelfths of a foot.

If numerical work is included with the geometrical some care is needed to prevent it becoming too difficult in the early stages.

### 2.4. Diagnosis of Attainment.

It is important that the teacher of an entrant class should acquire the maximum possible information about the attainments and attitudes of his pupils.

The necessity for this diagnosis and consequential remedial treatment was mentioned in 2·01 but was followed by suggestions for work new either in content or approach. From the early work on lines similar to those suggested the teacher will have gained a valuable insight into the extent of his pupils' mathematical equipment.

We now detail specific questions to which the teacher should try to find answers, and many of which he most certainly will have had in mind when planning that early work. Informal conversation with the children will provide much of the information required.

Chapter 4 treats many of the sections below in greater detail.

**2·41. Arithmetical attainment.**

(*a*) *Notation and conception of numbers*

How well does the pupil appreciate the way in which large numbers are written? Can he readily break them into units, tens, etc. and vice-versa? Can he write five hundred and four correctly as 504 and not as 5004?

(*b*) *Number bonds and mental reckoning*

(i) Has the pupil confidence in using addition–subtraction bonds? Does he tend to add in ones (or to count on for subtraction) either in general or in higher decade addition, e.g. $28+7=35$?

(ii) Has the pupil some mechanical knowledge of the multiplication–division table? If so, does he use his tables readily in problem situations in which they are appropriate? If not, does he reach the result by group counting or by counting in ones? Does he know his group sequences, e.g. 4, 8, 12, 16, 20, etc.? Does he appreciate the relationship between multiplication and division?

(*c*) *Four rules in number*

At what stage is the pupil in learning the procedure of the four rules for numbers greater than ten? If procedures are known, does the pupil know when to use them? (*b* ii). It is probable that some measure of remedial treatment is required by many pupils in the rules themselves or in their use; e.g. there may be some who add 4 thirty-sevens rather than perform a multiplication.

(*d*) *Money, weights and measures*

(i) Has the pupil a practical acquaintance with money, weights and measures? If so, is he skilful in counting money and using instruments for weighing and measuring?

Does he understand approximate measuring, e.g. to the nearest inch? Has he experience in estimating measures?

(ii) Can he deal by commonsense methods with small computations arising in his experience? Formal rules for written calculations are hardly of importance at this stage and much may be learned about a pupil's powers of dealing with numbers, as in (*a*) and (*b*) above, by observing the flexibility with which he deals with situations involving money and measures.

(iii) Has he had a realistic introduction to larger measures such as the mile, or to any of the more difficult concepts such as area?

(*e*) *Notation of fractions*

Has he learned to use the notation of fractions in familiar contexts, e.g. 4 oz. $=\frac{1}{4}$ lb.?

(*f*) *Other arithmetical knowledge*

Does he know anything about decimals, averages, factors or speeds?

(*g*) *Vocabulary*

Has he an adequate vocabulary for describing what he does in arithmetic, e.g. 'To find the cost of four pounds at 9½d. I multiplied 9½d. by 4'?

**2·42. Spatial attainment.** The spatial knowledge of the eleven year old is well worth more careful assessment than it generally receives. A child of eleven has, of necessity, adjusted himself to his many spatial experiences, but without adult intervention his ideas would be lacking in precision and generality. Information should be sought on the following points:

(*a*) *Vocabulary*

What is the extent of his vocabulary for describing spatial properties and relationships? Does he understand the words he uses? Does he know the words for the spatial concepts he recognises, e.g. *diagonally* for the corner to corner way across a field?

(*b*) *Position and direction*

Does he understand the need for methods of fixing position, as for buried treasure, fire hydrant, position of ship? Does he know what is meant by direction and how it is described, e.g. by mariner's compass?

(*c*) *Mensuration*

Has he acquired any knowledge of area and volume and their measurement?

(*d*) *Draughtsmanship*

Has he any skill in draughtsmanship? It may still often be found that boys have been given more geometrical experience than girls, particularly in the use of drawing instruments.

**2·43. Representation.** Primary school courses may include valuable experiences of representation to scale and in the diagrammatic representation of quantities which are not lengths, e.g. number in class, marks, etc. A few particular cases of graphs of statistics may also have been included. A pupil who has had this experience may give evidence of it in a variety of ways: does he suggest using a diagram to clarify a problem, or show initiative in collecting information?

**2·44. Conclusion.** Individual assessments of pupils in an entrant class are not enough; from them the teacher has to determine what he considers a body of attainment for his class as a whole, and which groups of pupils, if any, are so advanced or retarded in particular ways that special opportunities or treatment should be arranged for them.

Even if there are well-defined groups based on differences of attainment it may still be possible and desirable to bring all together for the initial work in such practical studies as those suggested in chapters 3 and 7 and in mathematical topics which are new to all. It is generally a difference in skill with the numbers which pupils can use which makes grouping necessary. A whole class may profitably combine in a topic which includes new work for some but revision from a new angle for others.

No group or individual should be made to feel left out or incapable of sharing in the activities and interests of a class to which he belongs, even if his contribution is a small one. (3·05 and 5·36.)

## CHAPTER 3

# THE PRACTICAL TOPIC AS A TEACHING METHOD FOR MATHEMATICS

### 3·0. Functions of the Practical Topic in learning Mathematics.

The freshness which a new context can give to mathematics teaching, whether or not the Mathematics itself is new to the pupil, has been exemplified in chapter 2, which deals with the work of the entrant to the Secondary Modern School.

We consider that at any stage in the Secondary Modern School the success of mathematics teaching depends not only on the careful consideration of the Mathematics itself but also on the context in which it is taught. The present chapter deals with the functions of appropriate practical experience as part of mathematics teaching.

**3·01.** Firstly it is against a background of experience that mathematical ideas take shape and notations gain meaning. At advanced levels the experience may itself be mathematical but most of the work appropriate to the modern school pupil is closely related to his practical experience. His general ideas are derived from consideration of many particular instances and the mathematical notations he learns are the language in which these ideas are expressed. For example, the important notion of ratio is an idea derived from many instances of comparison of many kinds of comparable quantities, and finally of numbers themselves.

**We feel strongly that all pupils need actual experience and experiment again and again as they make advance in mathematical expression and procedure. Verbal description of a situation is not enough, and mere learning of rules will fail to carry over into new contexts.**

Only the older abler pupils in Modern Schools are mature enough mathematically to pursue very far the study of Mathematics as an abstract system and even these benefit from frequent recourse to experience.

**3·02.** Secondly, the phrase in popular usage 'Applications of Mathematics' focuses attention on another aspect of the link between Mathematics and experience. It suggests, not the foundations of mathematical thinking, but the putting into action of mathematical ideas and procedures already established, in solving problems with a realistic setting.

**3·03.** Thirdly, there are many general topics which are best studied mathematically, or at least have many mathematical aspects. The economics of every-day life is an outstanding instance of a subject of social importance traditionally treated as part of the syllabus of arithmetic, and not otherwise included in the school curriculum. Air transport is a subject of interest to young people, but not a traditional part of any subject syllabus though it has contacts with History, Geography, Mathematics and Science. Both these topics, among others, are discussed in this chapter.

A practical topic which is included primarily for its own sake may also serve the functions described in 3·01 and 3·02; that is, it may be used as a source of new mathematical ideas, notations and procedures and as a field of application for those already established.

**3·04.** In choosing topics for study the teacher must not only consider the pupils' stage mathematically but also the maturity of their interests. Thus a consideration of the stamping of insurance cards is of interest to older pupils though arithmetically simple. Again the stamping of savings cards, and then changing them for certificates, is an activity for infants, but it is left to older pupils to deal with questions of interest on their savings and the appropriate tables for computing it.

**3·05.** The examples in the rest of this chapter are chosen to illustrate different types of practical topics and different ways of treating them as topics rather than to elaborate the paragraphs above. All the topics are presented as extending over a considerable period of time, but items from them might be used incidentally on occasion.

The organisation of the work so that there may be actual experience and collection of data by pupils is also described in some typical instances.

It is assumed that in using any practical topic as part of a mathematical scheme, teachers will pay due regard to its opportunities for the mathematical development and practice which are the subject of later chapters. No topic mentioned is necessarily so advanced that any pupil need be excluded from taking some part, though perhaps a humble one, in its study. It is however vital to successful study that the topic itself should be one which appears worth while to the pupils and to the teacher.

### 3·1. The Economics of Everyday Life; an example of a topic of social importance.

**3·11.** Topics of a financial kind are traditionally treated as part of an arithmetic syllabus. Their practical aspects are however generally dealt with by discussion only and as shortly as is consistent with learning how to do the computational work of problems set in the conventional verbal form. This leads, except with really able children, to books full of computation but to very little appreciation of practical issues unless these are matters of common experience like retail shopping.

The Mathematics involved in money topics is basically simple; that is, it consists of some combination of the four rules applied to money. The main mathematical advance in thought is to be found in the growth in value of a loan at simple or compound interest as the time of the loan is extended. Even the problems which are simple mathematically cannot be solved without a good acquaintance with the circumstances which give rise to them. For many children this can only be obtained by actual or simulated participation in money transactions. Even able children may fail to carry learning into future practice through lack of acquaintance with details of practical procedure.

**3·12.** Teachers who have faced the issues of teaching money topics as an education for citizenship take pains to see that their own knowledge of the practical background is detailed and up-to-date, a very considerable task, and then devise ways of managing classes of pupils so that they can master these details for themselves. This may involve duplicating mock forms, cheque books, savings books, tabulated material such as tables of interest or cost of motor vehicle licences, and so on, so that the pupil may have the necessary experience of dealing with the actual details of supposed transactions. Many of the better ways of managing classes for such activities depend on group and individual work with an element of dramatisation in it in which pupils are engaged on different jobs, some of them as it were as customers, and others behind the counter giving advice, cashing cheques, issuing licences etc. The difficulty of the jobs which individuals undertake can be varied by the numbers used and by the complexity of the situations involved.

**3·13.** Some of the notions of finance, for example rateable value, are not easily understood even if numbers are kept simple. Rateable value is more likely to be appreciated if treated in relation to a particular building and the rates as part of the householder's, or tradesman's, yearly commitments. Consideration of public finance may bring the difficulty of large sums of money and the standard forms for comparing them, £29,500,000 as £29·5m. or £$2{\cdot}95 \times 10^7$. Graphical representation of various kinds may be helpful in making comparisons and so on.

Some of the problems of citizenship may be examined with quite simple computation, others of necessity involve working beyond the less able. Forethought is essential; many well-meaning attempts to deal with the practical background fail because the number work proves unexpectedly heavy. The types of calculation which may be involved are discussed in 6·7.

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**3·14.** The following notes may be helpful:

(*a*) *Shopping*. The problems of the shopkeeper provide an advance in thought on those of the shopper commonly treated at the primary stage.

Hire purchase is well worth examining both for its social and its mathematical implications. It forms a good introduction to the idea that when credit is given and payment is delayed, the length of time involved is of importance in determining the amount of the payment. Credit sales may also be discussed.

(*b*) *Post Office*. The variety and intricacy of some of the transactions at the post office make this a practical project in itself with many aspects in advance of those treated at the primary stage. Savings bank deposits are a good particular example of payment of interest. The equivalence of $2\frac{1}{2}\%$ and 6d. in the £ is useful numerically. Also $2\frac{1}{2}\%$ per annum is $\frac{1}{2}$d. per £1 per month.

(*c*) *Simple and compound interest*. Many of the examples given in textbooks are unrealistic. Simple interest is generally applicable to short periods of time. Accumulation of a sum of money at compound interest is an important example of the more general problem of growth.

(*d*) *Profit per cent*. Commercially profit is generally reckoned on takings or selling price. Many textbooks work profit on cost price because of theoretical implications. The subject could well be treated as one of the shopkeeper's problems.

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(*e*) *Discount*. There are various types, trade discount, discount for cash, for quick sale, etc. The rate of discount is often expressed in the form of so much in the pound rather than as a percentage.

(*f*) *Insurance*. Again there are various types, personal insurance, property insurance, insurance against special risks such as a wet sports day, and so on. The idea of chance is involved; this can be discussed though it cannot generally be treated quantitatively. Simple examples of chance and of sharing a risk can be devised e.g. a sixpenny insurance for a number of parcels each worth £1 makes it possible to replace 1 in 40.

(*g*) *Income tax*. This is best treated by obtaining an official blank form from the local office and working out the various code numbers for a number of selected incomes and situations. Pupils are then sent to the nearest public reference library to consult the code tables. The time required to gain competence in completing such a form with fictitious amounts is not justified and can be more usefully employed. It is, however, of inestimable value to spend time discussing the completion of the form and the various ramifications that arise from it.

(*h*) *Wage and time sheets*. Sample sheets can be obtained from local firms and examples worked out. These usually include multiplication of small sums by numbers up to 48, fractional additions for overtime, piece rates, etc., as well as deductions for National Insurance, P.A.Y.E. etc. Such work can be regarded as a project of limited duration, say for a week or 3 or 4 lessons.

(*i*) *Foreign exchange*. This can be approached through planning a foreign holiday. It might also arise in discussing the services offered by a bank or from some comment in newspaper or broadcasting.

(*j*) *Additional possibilities*. Banking and cheques; local and national finance; the fluctuations of a stock or share. Domestic expenses such as gas and electricity. The building or buying of a house. Furnishing, etc.

**3·15.** We think it important that so far as possible the calculations involved in these studies should be in line with normal trade or professional practice; they might include for instance the use of ready reckoning devices such as:

(1) Avery scales in shops, in which a weighing shows the total cost for a given set of prices per pound. The making of a paper scale for various prices is a useful exercise. Other cost and quantity tables may also be constructed.
(2) Tables of costing in timber yards.
(3) Post office tables for national savings.
(4) Tables of insurance rates.

Calculations with £ s. d. should be fitted to the situation, which should be a real one and should not be introduced simply for the sake of elaborate reckoning (6·7). For example, decimalisation of £ s. d. is appropriate to a limited range of calculations. It is useful when dealing with percentages because of the division by 100, and the ease of obtaining an approximation to the shillings and pence. Percentage as a form of expression has a much wider context than that afforded by the economics of everyday life.

### 3·2. The Use of Environment ; rural examples.

The school's environment, whether in town or country, is always one factor in designing suitable courses of work. Sometimes the nature of the neighbourhood is so well-defined that it clearly determines the mathematical opportunities and the general knowledge and trends of interest of pupils. For example in a town there may be some dominating industry; or the town may be a port in which not only do port activities provide work for the adults of many families, but there is an outgoing of interest to countries overseas with their differing customs, climates, coinage, etc. In towns generally, however, the environmental interests are complex and many are too adult for school use.

**3·21. The rural school.** The school in rural surroundings is, therefore, chosen to exemplify the use of environmental opportunity. In such a school the Headmaster often solves a problem in the field of method, which arises from his desire to make the school a place where the pupil will react with his whole personality, by closely integrating its work with the life of the rural community. Such a Headmaster would for example view the local survey not as an end in itself but as a means of teaching. He might go further than this and constantly seek through the life and circumstances of the village the right 'hooks' on which to hang his curriculum.

**3·211.** For teachers of Mathematics it is easy to find applications of number, space and time in the rural environment. The concepts themselves and the language, verbal and symbolic, in which they are described will gain in depth and precision through use in well-understood circumstances. Some of the work will exercise only skills already mastered; it may for instance be based on actual figures from some friendly farm of costs and quantities of feeding stuffs, milk yields, and so on. These statistics may give material for orderly and graphical statements, and for purposeful straightforward calculations. The pitfall of calculation for its own sake will be avoided if there is a spirit of real enquiry and an informed interest in what is good farming practice.

When the atmosphere of the school is stimulating and encouraging, more and more interesting questions crop up. Consider for example:

'How many anemones are in this field? What are they worth? How are they marketed?'
'Which is the steepest hill in the parish?'
'How high is a fully grown elm and does its height depend on the soil?'
'What are the commonest weeds in the school field?'

In most cases answers can be found in more than one way : always the problem can be tackled first hand. Such an approach throws up problems of measuring, counting, computing, sampling, and the representation of elementary statistical analysis. The Mathematics used may involve little more than the four rules, averaging and pricing, all with suitable approximations, and scale drawing. Nevertheless, these first hand experiences will have great value if used to the full. The processes may be carried out in that atmosphere of enquiry and discussion which is natural to a truly practical approach. Because the events are real, it is easier for pupils to describe them. The linear measurements are directly made, thus forging a link between spatial and numerical perceptions; this is a vital fact upon which more advanced mathematical conceptions depend. Moreover, as the pupils select their own units, whether they use the traditional ones or not, these must be significant and appropriate to the situation; the degree of accuracy in the calculated result will be determined by their measurements. A computation may be shortened, or made in round numbers, but should in itself be completely accurate (6·27, 6·84, 6·85).

**3·212.** Among the mathematical concepts that always present difficulty are those of area and volume. A historical approach is linked closely with country pursuits. The Romans chose as unit of area the amount of land which a pair of oxen could plough in one day, from which came our acre. At first it was a definite shape but not a square. It was a furlong in length and one tenth as wide. Later in an English statute of early date the acre was given in more than one shape, e.g. 16 perches by 10 perches, thus making use of the idea of conservation of area. A pupil needs an ample range of experience to accustom himself to this idea of equality of areas of different shapes. A local map of say 25″ to 1 mile may well show contrasting shapes for fields of about the same acreage, and such examples help to illustrate the point. Examples may be available in field and garden to reinforce the work of the classroom (6·522).

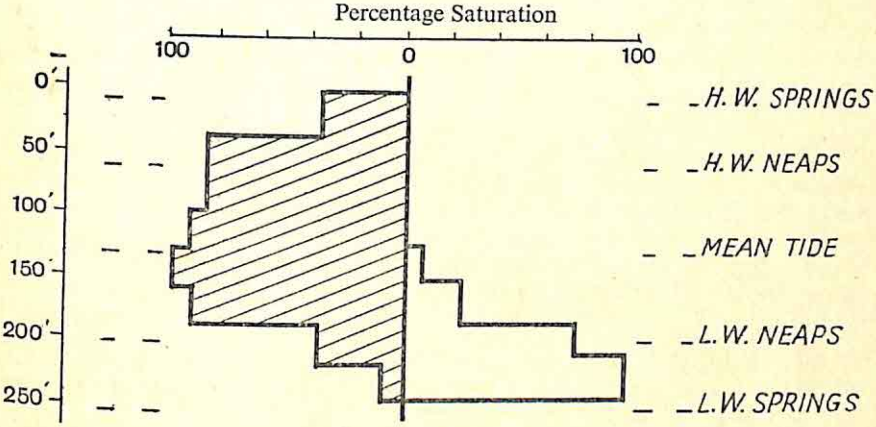
**3·213.** Interesting examples of measures for particular purposes (6·813) are often to be found in the countryside, for example a 'square of thatch'. Special rules for measuring particular shapes are also used. For example, a more convenient rule for the cross section of a cylindrical silo is $\cdot 08(\text{circumference})^2$, rather than the familiar $\pi r^2$ because the circumference can readily be measured by passing a string round the silo whereas it is harder to measure the radius of a circle whose centre is not known and is not easily accessible.

The experienced countryman's ability to estimate lengths, areas, weights, etc. may arouse interest and be compared with the boys' and girls' own estimates or laborious measurements. The way in which estimates are made is also worth enquiry; sometimes a rule of thumb may be involved, at others it may be simply a matter of such rich experience of measures that an easily comparable value is ready to mind, or the complex shape of a haystack may be so familiar that it can be thought of as a whole.

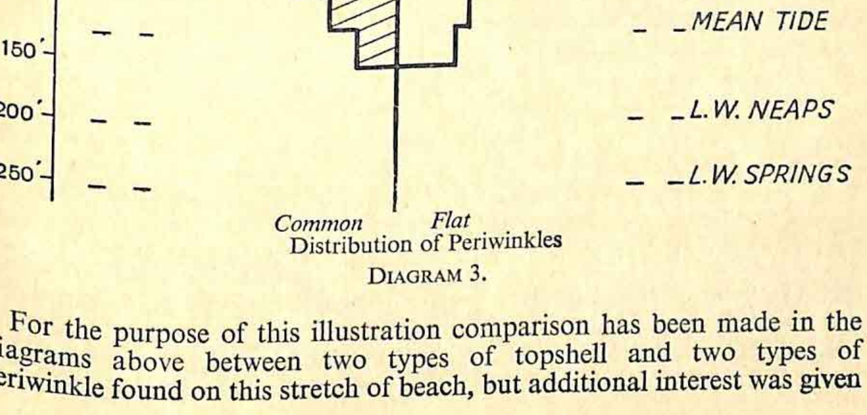
**3·22. Fieldwork.** Perhaps the most satisfying aspect of practical Mathematics is for the child to see or produce evidence of something achieved. The planning of a piece of fieldwork, the collection of data and the presentation of these data can be stimulating to all concerned. Two examples of field exercises within the scope of quite average youngsters are now given, the one statistical in nature and the other involving some development of geometrical concepts together with motion.

**3·221. A problem of distribution.** Although the following investigation was not performed by boys and girls it is given as typical of a practical activity within their capabilities. Its general method might be applied to other natural distributions.

Information was required about the distribution of shells of various types on a shelving rock beach. Two parallel strings, 20 feet apart, were placed the length of the beach from the top down to low water level. The strings were graduated every 10 feet and the area of beach thus divided into strips, 20 feet wide and 10 feet deep. Ten wire frames, each one foot square, were thrown at random on to each strip in turn, and a count made of the number of frames which contained specified molluscs. In this way, each species was given a score, ranging from 0 to 10, for each strip throughout the length of the beach.



*Flat* *Grey*
Distribution of Topshells



*Common* *Flat*
Distribution of Periwinkles

DIAGRAM 3.

For the purpose of this illustration comparison has been made in the diagrams above between two types of topshell and two types of periwinkle found on this stretch of beach, but additional interest was given to the results by the fact that a third type of topshell and a third type of periwinkle were found in small numbers, the first at or below low water and the second in the splash area at or above high tide level.

The frequency diagrams (Diagram 3) have for purposes of comparison been placed on either side of a line showing distance up the beach, instead of side by side with this line drawn across the page separately for each kind of shell as is more usual when comparison is not involved.

The scores, from which the above diagrams were constructed, were given for every 10 feet down the beach, but in the diagram, the results have been grouped every 30 feet. The decision to re-group is an arbitrary one, and is governed by many factors. Perhaps the following table, in which the scores have been grouped in intervals of 10, 20, 30, 40, 50 and 60 feet will bring out some of these factors.

| Distance down shore (feet) | SCORES (flat topshells) | | | | | | |
|---|---|---|---|---|---|---|---|
| | 10's | 20's | 30's | 40's | 50's | 60's | |
| 0 | | | | | | | |
| 10 | 0 | 1 | | | | | H. W. SPRINGS |
| 20 | 1 | | 11 | 21 | | | |
| 30 | 10 | 20 | | | 29 | 37 | |
| 40 | 10 | | | | | | |
| 50 | 8 | 16 | 26 | | | | |
| 60 | 8 | | | 33 | | | H. W. NEAPS |
| 70 | 8 | 17 | | | | | |
| 80 | 9 | | 26 | | 44 | | |
| 90 | 9 | 19 | | | | 54 | |
| 100 | 10 | | | 37 | | | |
| 110 | 9 | 18 | 28 | | | | |
| 120 | 9 | | | | | | |
| 130 | 10 | 20 | | | 48 | | MEAN TIDE LEVEL |
| 140 | 10 | | 30 | 40 | | | |
| 150 | 10 | 20 | | | | 58 | |
| 160 | 10 | | | | | | |
| 170 | 9 | 18 | 28 | | | | |
| 180 | 9 | | | 28 | 38 | | |
| 190 | 6 | 10 | | | | | |
| 200 | 4 | | 12 | | | | L. W. NEAPS |
| 210 | 2 | 4 | | | | 15 | |
| 220 | 2 | | | 5 | 5 | | |
| 230 | 1 | 1 | 3 | | | | |
| 240 | 0 | | | | | | |
| 250 | 0 | | | | | | L. W. SPRINGS |

The groupings in intervals of 10 and 20 feet contain fluctuations which might be genuine, but are more likely to be the result of the sampling method used, and the small numbers involved. The groupings in intervals of 50 and 60 feet would appear to have reduced the number of intervals too much to be of much value, and the choice of interval would appear to be between the remaining two shown. The decision on size of interval for such a distribution should be the result of much useful discussion.

Once the information obtained in the experiment has been reduced to the most useful diagrammatic form, the interpretation of the diagrams presents more opportunity for discussion. Study of the above diagrams, for example, would give rise to several conclusions, of which a few might be

(*a*) Topshells are more plentiful than periwinkles on this beach.
(*b*) Flat topshells are distributed throughout the whole beach and probably not below low water spring level.
(*c*) Grey topshells do not appear to reach beyond the mean tide level, and probably exist below low water spring level to a similar extent.
(*d*) Flat periwinkles tend to be somewhat lower on the beach than common ones.

Another source of discussion, and possibly of comparative experiment, might be the method of sampling used. It might have been possible to score by counting all the shells of a species found in each of the random castings of the wire frames, or to have distributed the frames according to a pre-arranged pattern instead of by chance. Again more frames could have been used with or without significant differences in the conclusions drawn from the distributions.

**3·222. To estimate the flow, say in gallons per minute, of a local river.** A solution will call for the concept that:

(Volume flowing in a given time) = (area of cross-section of river) × (length occupied by water passing in that time.)

The pupils will examine the site, discuss the problem, and decide what measurements are necessary and how they can be made. There may be, for instance, a bridge across a typical section of the river. From it, depths could be sounded at, say, one foot intervals, and the irregular section plotted to scale, on squared paper, and its area computed. Then the average rate of flow could be determined by timing objects floating over a fixed measured distance, using a stop watch. There will inevitably arise discussions as to possible refinements: for example, as to drag, whether to neglect any markedly differing time reading, and so on. Such discussion can be most profitable: at the very least pupils should never believe that the flow is known; but they should feel satisfied that a reasonable estimate has been made. There will be difficulties in the final computations, and much careful consideration of units. When found, the flow can be related to the water needed daily by the community. This may call for further investigations; it will certainly involve fresh demands on the concept of conservation applied to the flowing water. This experiment has been tried and it has been extended by taking samples at more than one point in the same river and so comparing rates of flow from different sets of data.

**3·223.** The examples described in the last paragraphs are only two from the very rich field offered by a rural environment. Opportunities for combining Mathematics with investigations of biological and geographical interest are particularly plentiful in a country school.

## 3·3. Topics of Contemporary Importance and Interest.

Young people are keenly interested in the world around them and man's inventions and discoveries. Many of these have technical and quantitative aspects, which call for mathematical description. From among these we have selected air transport with which to illustrate the kind of mathematical opportunity arising from such a topical study.

In this section we show mainly in tabular form the sort of organisation of material which might be made, and later discuss the way in which a class might work on it. The range of ideas and applications arising from movement through the air, which is three-dimensional, is wider and much richer in human interest, content and significance than can be found in most traditional school Mathematics. The topic of air navigation is treated more technically in 7·2.

**3·31.** The following suggestions of the possibilities of this topic vary greatly in difficulty; which of them can be used with any particular class must depend on the interest and knowledge of the teacher, and the opportunities for the pupil to see aircraft and collect data about air travel. Pupils who are interested will themselves seek information outside the school, but at least a nucleus must be readily available when the study begins. The pupils themselves will suggest many of the lines on which it should be pursued and there is room for individual and group divergences.

### Air Travel as a Practical Topic

| (*a*) Social and Personal Aspects | | Mathematical Opportunities |
|---|---|---|
| Journeys | timetables | Distances.<br>24 hour clock, local times, time for a journey.<br>Accessibility of airfield. |
| | cost | Fares, comparison with other means of travel.<br>Foreign currency. |
| | luggage | Weight allowed, estimate of items to fill a suitcase, sportsgear. |
| Freight regulations | | Problems of size and weight, fuel load, safe load. |
| Accident rates | | Statistics. |
| (*b*) Historical Aspects | | |
| (Development of air transport)<br>Types of plane, 'spotting' | | Spatial appreciation, vocabulary for description (bi-plane, monoplane, helicopter, sweep back, dihedral angle etc.).<br>Reading plans of structures, model-making.<br>Balloons, airships, 'Heavier than air'. |

| | |
|---|---|
| Propulsion and speed changes | Speed in general (an important digression).<br>Statistics and graphs of speeds attained, of heights attained.<br>'Faster than sound'.<br>'How long would it take a jet to cross the playground, to fly round the earth? etc.' |
| Use of Transport | Graphs to show changes in mileage, freights, etc. |
| Future Development | Possibilities of rockets and space travel. |

## (*c*) Geographical and Geometrical Aspects. (7·2.)

| | |
|---|---|
| Routes and Maps | Distortion of global distances on a plane map.<br>Lengths of journey on a globe.<br>Rhumb lines and great circle bearings. |
| Placing of Airports | Distances between ports, refuelling.<br>Length of runways. |
| Navigation | Representation of vector quantities, (displacements, velocities, etc.).<br>Allowing for wind speed.<br>Determining position, dead reckoning, checking by position lines.<br>Pilot's log of flight.<br>Gliding.<br>Language; bearing, backbearing, course, vertical, angle of attack, angle of glide, angle of climb, etc. |
| Time | Time in relation to longitude.<br>Zones of time. |
| Weather | Weather data with graphical representation.<br>Visibility scale, Beaufort scale. |

## (*d*) Mechanical and Scientific Aspects

| | |
|---|---|
| Miscellaneous | Reading instruments.<br>Revs. per second.<br>Movements of under-carriage. |
| Engines | Horse power, comparisons with land travel. |
| Pressures, etc. | Air pressure at different heights, air conditioning. |
| Parachutes | Superficial area, retardation etc. |

**3·32.** The classification above is only intended to help the teacher and is somewhat arbitrary. Many of the suggestions might have been included under more than one heading; the list is far from exhaustive. It is in fact the sort of list a teacher should make for himself when his class is embarking on a practical topic, whether he is the only member of staff concerned, or one of a team.

**3·321.** If possible the pupils should suggest what they would like to find out about air travel, and they should do this freely without being limited to the mathematical or other aspects. The teacher, with the help of the class if possible, should group the enquiries in some way and let each pupil join a particular group, and work with that group until he has definitely achieved something in it: made a map, collected some statistics, acquired some knowledge which he can pass on, reached a point where new lines of enquiry are opening up, etc.

For example, one group might try to gain information about the size of various types of aircraft, and might mark out on the playing field rectangles to represent the necessary size of hangar for each and draw such rectangles to scale on squared paper. A second group might study the carrying capacity and internal seating arrangements of aircraft, perhaps using chairs in the hall to make this clear to themselves before drawing a plan. A third group might be interested in the weights carried, and the stowage of luggage. A fourth group might consider speeds, timing schedules, and fuelling. A fifth might make maps of routes and learn about the measuring of distance on a sphere. A sixth might pay attention to methods of propulsion, buoyancy and stability in flight.

**3·322.** Each group should know where books likely to help them are to be found. It may be well to apportion some of the general books to each group to begin with, as well as to move the classroom furniture into blocks so that each group works in a well-defined space, in which it keeps all its materials, including supplies of paper, instruments, etc. It is not necessary that all members of a group should do the same work, or that the composition of groups should be unchanged throughout the time the topic is being studied, but organisation on group lines enables the teacher to supervise the work, minimises the amount of movement necessary in fetching things, and makes the best use of available reference books, etc.

**3·323.** As he moves from group to group the teacher will find that he has to add to his list of mathematical opportunities, and that he and the pupils have to meet and solve new and unforeseen problems. Members of a group and individuals should be able to explain what has been done and found out, and to pose their own questions. At the end of the topic, or at intervals during its progress, some form of open discussion or display should be used to assemble the work of the various groups and show what the work of the class as a whole has been.

**3·324.** It may also be desirable to devise some record of progress such as a tabulated sheet on which each pupil ticks off or records a note of each small 'job' which he contributes to the whole. Such a record might include the unofficial homework to which a topic of this sort gives rise and information brought in from outside.

**3·325.** If the need for a new mathematical technique arises, e.g. distances to be measured along great or small circles of a sphere, it will be well to show first the individual pupil, or a group, as much as is needed for the matter in hand, and later to take up the question with the class as a whole either apart from or as arising out of the topic.

### 3·4. The Relation of Mathematics to other Subjects of the Curriculum.

There are parts of other subjects of the school curriculum which are illuminated and widened if studied mathematically, but their place in the course must be determined by consultation between members of the staff, and not by reference to the pupils' progress in one subject only. Our views on the kind of co-operation desirable with members of staff teaching allied subjects, particularly Science, Handicraft and Geography will be found in other sections; see 3·223, 3·31 and 3·51 in this chapter and 5·321, 5·331, 7·214, 7·3 etc.

We give below some details of work suitable for pupils studying Gardening and Housecraft in school. If both these subjects are taken in the same school, but in separate classes for boys and girls, it should be noted that cropping in the garden, its timing, quality and quantity, is of importance to training in housecraft; that work in mixtures and soil analysis is comparable with that on recipes in cooking and solutions for house-cleaning; the placing of rows in a plot is somewhat similar to problems which occur in decorative crafts. But if the mathematical work is taken together it would be well for the girls to take a real part, if a small one, in the gardening, and for the boys to do some housecraft, or be otherwise interested in the work suggested under that heading.

**3·41. Mathematics and the school garden.** Work in the school garden offers a great variety of experience of using numbers, measures, diagrams, etc. and if the gardening includes organised experiment it will give opportunity for training in precision in measurement and careful recording of statistics. The numerical examples included below are drawn from some actual experiments performed in schools. They do not in any way form a syllabus. Most of them depend for their occurrence on the work undertaken in the gardening course and they are only given in tabular form to show the range of opportunity and the inclusion of measures of a conventional and unconventional kind. The work on, say, weather or surveying or percentage analysis might be developed in the mathematics course beyond the immediate needs of the gardening.

*Numbers.*

(*a*) Counts and efficient techniques of counting e.g. 250 peas, or 100 or 200 seeds for planting.

(*b*) The four rules
- e.g. (i) 20 tons per acre is $x$ lb. per plot of 24 sq. yd.
- (ii) adding simple fractions of 1 lb. in cropping totals
  e.g. $4\frac{3}{4} + 6\frac{1}{2} + 9\frac{3}{4} + 7\frac{3}{4}$

| | | |
|---|---|---|
| *Measures.* | Lengths. | Depths of soil and of planting; heights of plants; distance apart of rows, of plants; lengths of drills; dimensions of plot with suitable placing of rows, e.g. the dimensions of each of a series of plots having 3 rows, 30 in. apart taking 24 potatoes 16 in. apart, and allowing 16 in. at each end of a row. |
| | Areas. | Areas of plots.<br>Unit surface area as a measure in using fertilisers etc., and in taking equal volumes of soil.<br>Riddle mesh in sorting potatoes (see weight). |
| | Volumes and weights. | Generally in connection with fertilisers or cropping, a great variety of practical measures may be used such as<br>(i) 3 good barrow loads.<br>(ii) 20 tons per acre is 9·2 lb. per square yard.<br>(iii) 1 fork-full of manure (12 inch fork) per yard of drill.<br>(iv) $\frac{3}{4}$ pailful per square yard.<br>(v) 2 oz. per square yard, or per yard of drill.<br>(vi) $\frac{1}{4}$ teaspoonful per plant.<br>(vii) $\frac{1}{2}$ gallon of creosote.<br>(viii) 1 ounce will sow 150 feet.<br>(ix) Both count and weigh 250 peas for sowing.<br>(x) Classifying seed potatoes by weight into classes, ($1\frac{1}{2}$ to 2) ounces, ($2\frac{1}{2}$ to 3) ounces, ($3\frac{1}{2}$ to 4) ounces.<br>(xi) Classifying potato crop by weight (and size by riddle) into 'eating potatoes', 'wares', 'seed potatoes', ($1\frac{1}{2}$ to 3) ounces 'pig potatoes', 'chats'. |
| *Time* | | Dates recorded of planting, germination, cropping, etc. Days between dates. |
| *Weather, etc.* | | Records of temperature, rainfall, drought, wind directions.<br>Position of garden, height above sea level, slope and aspect. |
| *Geometry.* | | Survey of garden, including slope.<br>Planning a rectangular plot.<br>Marking a drill in a straight line.<br>Direction of wind, of drill etc. |
| *Percentages.* | | Results may be given as percentages, e.g. germination per 100 seeds sown.<br>Soil analysis, percentage of clay, of sand, etc.<br>Pest control, e.g. $\frac{1}{2}$ teaspoonful of 4% calomel dust per plant. |

This list is suggestive, but of course not exhaustive of the mathematical implications and opportunities of gardening work. If experimental work is undertaken problems of sampling and recording statistics will also occur.

**3·42. Mathematics and Housecraft.** Housecraft involves a great variety of activities which call for careful planning or which are based on scientific principles. The list below of experiences using numbers, measures, etc. is again not a syllabus but an analysis of the kind of numerical and spatial work which might occur in a housecraft course. The value of this as Mathematics depends to a large extent on the practical studies which are going on concurrently. There are, however, some which might find their practical background in boys' and girls' own home experiences and supplementary information required can be brought to the ordinary classroom. For instance in dealing with home furnishing and decoration, expert advice may be needed on questions of quality, taste and craftsmanship but costs and quantities can be dealt with as a part of arithmetic provided the necessary measurements can be taken.

In 3·421 the subject of nutrition, essentially a technical housecraft study, is set out at some length to show how its study might depend on co-operation with Mathematics. In 3·422 some notes are supplied of a study with a homely background undertaken mainly in the Mathematics classroom. Some of the general opportunities for using numbers and measures are set out below.

| | |
|---|---|
| *Numbers.* | Counts in stocktaking.<br>Four rules . . . wide range of use in measuring and costing, e.g. length of carpet required for 14 stairs and small landing 3½ feet long if treads are 10¼ in. and risers 5½ in. |
| *Lengths and areas.* | Widths of materials (36″, 54″, 27″, 32″), of lino etc.<br>Placing patterns; measures in dressmaking.<br>Idea of area from dusting, scrubbing, ironing, etc.<br>Shelf area, and capacity of oven or refrigerator in terms of shelf area and height. |
| *Weight, capacity and density.* | Gallons in washtubs, etc. pints in kettles, etc.<br>Weights of different substances per spoonful.<br>Weights of different substances per cupful.<br>Size of packet containing 1 lb. different goods. |

| *Standard cans* | *Average net wt.* | *Capacity in cups.* |
|---|---|---|
| e.g. No. 1 | 11 oz. | 1⅓ |
| No. 3 | 33 oz. | 4 |
| No. 5 | 3 lb. 8 oz. | 7 |
| No. 7 | 6 lb. 10 oz. | 13 |

Quantities for class or school meal from data below.
Soup ½ pint per head for main course.
¼ pint per head as introductory course.
Flour 1½ oz. per head for puddings or pastries.
Potatoes 6–8 oz. per head uncooked.
Green vegetable 4 oz. per head.
⅓ to ½ pint milk pudding per head.
Sausages (6 to 8 to the pound), or meat according to cost, at a chosen cost per head.

| | |
|---|---|
| *Time.* | Times for cooking; when to light oven if allowing 20 min. to the pound.<br>Time to boil the kettle, cost of cooking, etc. |
| *Temperature.* | Oven thermometers and gauges.<br>Degrees Fahrenheit to Centigrade and vice versa in connection with certain recipes for candy and fruit preservation.<br>Pressure cooker. |
| *Geometry.* | Packing a cupboard, shapes and area.<br>Dressmaking, layouts, cut on the cross (diagonal of square framework), adaptations.<br>Planning kitchen, economy of movement. |
| *Ratio, percentage proportion.* | Adapting recipes to size of family, patterns to size in dressmaking, etc.<br>Cleaning and stiffening solutions, e.g. dilute in ratio 1 to 3.<br>Use of water softening agents. |
| *Money.* | Costs of meals, furnishings, decoration, heating (including allowing for cooking costs or heating, drying and ironing in laundry).<br>Relative cost of bought and home-made articles.<br>Gas and electricity accounts. |

Many of the above ideas may be approached through short projects covering some aspect of the running of a household and the economics of everyday life.

**3·421.** The study of nutrition involves some difficult ideas; it is the business of the housecraft teachers to decide whether any group of girls is able to profit from them. The computations involved are not themselves too difficult provided the quantities to which they are applied are meaningful. They are however apt to be cumbersome, so that the housecraft department may hesitate to embark on them without the co-operation of teachers of Mathematics. On the other hand some of the units involved are rather unusual and the mathematics department may well shrink from using them without technical help.

Thus rates of using energy may be given in one of two ways: calories per kilogram bodyweight per hour, or, calories per square metre body surface per hour. The idea of a calorie as a meaningful unit has first to be accepted and there are ways of estimating body surface. The calorie values of different foodstuffs and the calorie expenditure in different activities are given in standard tables e.g.

100 cal. from $1\frac{1}{2}$ oz of white bread (or ice cream)
or 4·23 oz. of potatoes
or $\frac{1}{4}$ pint of milk
or $1\frac{1}{2}$ eggs.
Walking at 4 m.p.h. uses 3·4 cal. per kg. per hour.
at 3 m.p.h. uses 2·0 cal. per kg. per hour.
Running uses 7·0 cal. per kg. per hour.

From such data examples might be worked out of expenditure of energy and its replacement.

Problems from the study of nutrition can be carried a stage further into the working out of a balanced diet which will supply not only the required number of calories but the requisite food constituents, vitamins, etc. When the pupils have become familiar with the function of different foods which are rich in proteins, carbohydrates etc. interesting experiments can be made in drawing up menus supplying the requisite quantities of each element. For this the use of tables of analysis of common foodstuffs is necessary. The tables generally show protein etc. as percentages.

The problem can again be carried further if a maximum daily or weekly food budget is suggested.

Although it is unlikely that a housewife would make such calculations in planning the food for her family, to study the problem numerically is likely to emphasise the kinds of consideration which are important. In particular a girl will realise that bulk is not a measure of food value and that the cheaper carbohydrate foods have to be combined with the protein and vitamin-rich foods which judged by bulk or weight are highly priced.

**3·422.** Boys and girls can study some topics connected with an ordinary household in the mathematics classroom with only occasional reference to the housecraft expert. Such topics might be suggested by the boys and girls themselves but could perhaps include making lists of the household tasks of mother and daughter, father and son. The lists would overlap in some respects. From them might be chosen a study of, say, lighting and heating in the home. Advice might then have to be sought from both science and housecraft departments. Amongst points dealt with within such a topic would be:

Electricity, gas and coal bills.
The units used in measuring gas and electricity.
Reading meters (with class models of meter dials).
The electricity (or gas) used for different jobs; boiling a kettle, cooking a cake, doing the laundry. Discussion of how to measure this, with actual experiment outside the classroom in the home or housecraft room.

## 3·5. Vocational Interests.

Towards the end of the school course when pupils are beginning to think in terms of obtaining employment vocational interests may play an increasing part in gaining their co-operation and effort in mastering mathematical ideas and skills.

What particular vocations are thus used in a school will largely depend on local conditions. A mining district, or an agricultural one, or a town with one predominant industry will almost necessarily provide vocational courses. Where many of the boys are likely to be employed in the building trades, or in electrical work, or machine shop engineering their work at school can be linked up with the Trade Courses of the Technical Colleges, and teachers will find suggestions as to how this may be done in the Association's *Mathematics in Technical Colleges*, but it is important that the school should not confine its attention to one trade course, but should make its pupils aware that the Mathematics they study has a variety of applications and uses.

**3·501.** For instance both boys and girls can be interested in work with a mechanical bias, such as the gearing and mechanisms of simple machines: the bicycle, the sewing machine, the washing machine, accounting machines, looms, pulley and tackle. One or more such machines can be examined or even taken to pieces in the classroom and form the basis of mathematical study for a period of time (7·4).

Quantities too can be studied; the count of yarns in a textile area, the reams of paper for a printing works, tare of wagons, head of water at a dam, measurement of brick-work, area and frontage of a building site, etc. All these are adult uses of measurement which can be of interest to the pupil looking forward to entering the adult world, but he should meet them in a real situation, such as a visit to the factory or the dam or the building site, and only afterwards make calculations or do other work in connection with them. Teachers will find that when pupils are taken on a school visit very few of them will notice such mathematical details unless they are helped to do so, but that they can be interested in this aspect if there is an adequate follow-up, and books with similar data are available. This provision of books is a difficulty; mere arithmetic text-books with invented examples do not really meet the case, and the teacher needs to collect trade pamphlets and to use annual publications as reference books.

The use of graphical records, and of automatic measurements on a dial, or otherwise, provide other mathematical opportunities of a vocational nature.

**3·502.** Sometimes a whole form is preparing for the same or a similar vocation, e.g., there is a commercial form, or a pre-nursing form. In such cases the vocational interest may play a large part in the planning of the course, but it should be borne in mind that a school is not only training its girls as clerks and nurses, but also as potential housewives, as prospective citizens, and as women with hobbies and leisure time pursuits, and that mathematical applications can be found, and will indeed be required in all these spheres, so that the course planned should not be narrowly vocational. Similarly boys must also be educated as fathers and citizens as well as wage earners.

**3·503.** Older boys and girls may well undertake the study of topics individually, or in twos and threes; it is not necessary that all members of a class should be engaged on the same topic. If pupils have become accustomed to working in groups on different parts of a class topic, and to searching out information from books and other sources, they will respond readily to the idea of choosing a topic, each for himself, and working at it section by section so as to present the work in some finished form. As with the group work the teacher should supervise and assist all and should devise some system of recording progress, for example an entry made of the name of a book consulted, a map drawn, a page of statistics completed etc.; he must also be ready to advise where information should be sought or material for study can be found; often he can suggest improvements in the way the material is being used and point out new lines of enquiry into its implications; he can also influence the form in which the results are presented.

**3·504.** The Mathematics of many vocational studies will be found under other headings. We add below some suggestions for the pre-nursing course based on enquiries made at a training hospital as to what knowledge was likely to be useful to the student-nurse.

**3·51. Mathematics and Nursing.** The syllabus of Mathematics for pre-nursing courses differs very little in its basic requirements from any normal course of arithmetic for a pupil of average ability, though some bias would be desirable in the final year of a four-year course. Particular stress is laid by the hospitals on mental calculations, the ability to measure accurately, and a good working knowledge of the metric system. The syllabus must include mensuration, the units used in pharmacy, simple proportion, percentages and graphs; it is important to know the kinds of calculation that a nurse may have to carry out in connection with her job as she must be equipped to do these speedily and accurately.

The Mathematics suitable for this course is essentially of a practical nature, and opportunity must be given for plenty of practice, particularly in measuring. The link between the Physics and the Mathematics is here especially strong, and if some of the mathematics lessons can take place in the science laboratory so much the better (5·331, 7·4). Failing this thermometers, weights and measures and other equipment should be available in the classroom. Probably here more than anywhere else in the school, it is the experience of handling weights and measures which is valuable, rather than merely calculating, although of course this must not be neglected.

The suggestions below adapted to nursing and suitable for Modern Schools are set out under mathematical headings.

**3·511.** *Fractions.* A thorough understanding of fractions is necessary, and to this end problems necessitating drawing, paper cutting and paper folding should be used to the full. Particular reference should be made to a fraction of a fraction e.g. $\frac{1}{2}$ of $\frac{1}{4}$. This is necessary when it is desired to give $\frac{1}{8}$ gr. of a particular drug and the tablets are only available in $\frac{1}{4}$ gr. The difference between $\frac{1}{2}$ of $\frac{1}{4}$, and twice $\frac{1}{4}$ cannot be too firmly stressed. Revision of the four rules is necessary for future calculations. A topic of interest which largely depends on fractions and decimals, is the working out of a diet sheet. Here the calorific value of food is given in the following way: $1\frac{3}{4}$ ozs. of stewed beef yields 100 cal., or 1 gramme of protein gives 4·1 calories (3·421).

**3·512.** *Decimals and Metric System.* A detailed knowledge of the metric system is indispensable for the prospective nurse, and since this depends entirely on the decimal the two are inseparable in this context. Familiarity with the metric system should be acquired by practical handling of the metre, litre, gramme etc. (5·331, 6·812). A collection might here be made of household labels which bear a double weight or measure e.g. 8 oz. or 227 grm. on a biscuit tin. Acquaintance with phrases such as '·5 mg. in ·5 ml.' is also useful. The importance of accuracy in measurement should be stressed in this part of the work, and answers should be given 'correct to one place of decimals', or to whatever number of significant figures the data allow (6·84). In particular, small units should be used. Finding area and volume in metric units is important and the connection between the metre, litre and gramme should be emphasised both by actual demonstrations, and a detailed study of its history. 1 cubic centimetre and 1 millilitre are two names for the same capacity or volume. This can be verified by filling a hollow cubic centimetre with sand, and tipping it into a measuring jar marked in millilitres. Syringes are now marked in millilitres, but were formerly marked in cubic centimetres; both are in use at the present time. The volume of various vessels, such as medicine glasses as well as the more common household utensils, may also be found by filling with sand.

**3·513.** *Tables.*

*Apothecaries' Weights*
20 grains = 1 scruple
3 scruples = 1 drachm
8 drachms = 1 ounce
12 ounces = 1 pound

The small weights of this table are used in mixing medicines, but the pounds and ounces are not much used. When drugs are being bought Avoirdupois weights are used; the Avoirdupois pounds and ounces are larger than those in the scale of Apothecaries' weights.
1 pound Apothecaries = 5760 grains. 1 pound Avoirdupois = 7000 grains.

*Apothecaries' Fluid Measure*
60 minims = 1 fluid drachm
8 drachms = 1 fluid ounce
20 fluid ounces = 1 pint
8 pints = 1 gallon

1 drachm = 1 teaspoonful
2 drachms = 1 dessertspoonful
4 drachms = 1 tablespoonful
2 ounces = 1 wineglassful

Details such as the above may be found in reference books, but teachers should check for themselves which units are in common use, and how they are used.

**3·514.** *Percentages.* Little new work will be necessary here, but some revision must be done and the meaning of such phrases as a 5% solution should be understood. It will also be necessary to interpret the percentages which occur in Public Health statistics.

**3·515.** *Ratio.* The nurse's main use for ratio will occur in the dilution of solutions and lotions when she will have for example to make up a solution of 1 in 60 from the standard 1 in 20. The work here can be practical as well as theoretical and several household parallels can be found, e.g. making up squashes, starches and disinfectants. The strength of a solution can be demonstrated by the use of ink, and the fact that 1 in 10 is stronger that 1 in 15 can very quickly be seen.

**3·516.** *Graphs.* Graphical work will assist in most topics in this course, but from the hospital's point of view the most important aspect is the interpretation and the plotting of a temperature chart. Conversion graphs converting degrees Centigrade to degrees Fahrenheit, yards to metres, Kilograms to pounds etc. should be understood and used. Graphs may also be drawn to show the average growth of a child. Graphs of population and health statistics will probably be used in the course.

**3·517.** *Use of Formulae.* Some work on the use of simple mathematical formulae is essential since the nurse in training is presented with several involved formulae which she is expected to be able to use. In school, formulae for the surface area and volume of some of the regular solids might be found, together with the formula for converting degrees Centigrade to Fahrenheit etc. (6·313, 6·316, 6·317, 6·35).

**3·518.** Other items which may find a place in the course if time permits include:

(*a*) Cubic capacity of domestic rooms, classrooms and wards; comparison with Public Health requirements.
(*b*) The mechanics of pulleys and levers.
(*c*) Density and specific gravity.
(*d*) Archimedes' principle.

Since all the work is directed towards a particular end, it is probably unnecessary to introduce extraneous topics. However one which could be used is detailed below.

*A day in the life of a toddler*. (a) Child's timetable, showing the time devoted to sleeping, playing, etc. (b) Calorific contents of meals. (c) Proportions required for making milk drink and orange juice; the problem of dilution also occurs in preparing antiseptic for washing. (d) Visit to clinic; collect information regarding height and weight of toddlers; present in graphical form.

### 3·6. Conclusion.

In this chapter we have dealt under a number of separate headings with the learning of Mathematics through the study of real situations which interest the pupils not primarily because of their mathematical content. We feel strongly that the teaching of Mathematics has often failed just because the teacher's interest or judgment has been allowed to dominate, and we plead that much more effort should be devoted to leading the pupil to an interest in Mathematics *through his other interests*. That is why this chapter has been placed where it is. Further topics which are more obviously and closely related to Mathematics will be found in chapter 7.

Mathematics teachers will need to include in their reading many books not usually to be found on the mathematics shelves of a library.

## CHAPTER 4

# REVISION AND REMEDIAL WORK

### 4·0. The Differing Functions of Revision and Remedial Work.

Revision by planned practice and by making new uses of mathematical procedures which have been learned is an accepted part of teaching method in Mathematics. We think however that it may be helpful to teachers to draw a distinction between this normal process of recapitulation and a type of treatment which may be needed by many modern school pupils which we call remedial treatment.

Revision is for matter more or less recently broached but not yet mastered. Remedy is required when matter far behind the apparent growing point of knowledge has been imperfectly assimilated or even missed out altogether. Gaps in knowledge form a class by themselves, but as a source of misunderstanding they are conveniently considered with remedial treatment.

**4·01.** We consider it most important that teachers in Secondary Schools should avoid the acceptance of any particular body of knowledge as 'primary school work' and therefore necessarily only the subject of revision or remedial treatment. Chapter 2 deals with the diagnosis of attainment, which will have to be accepted at many levels.

**4·02.** No harm is done if revision is approached remedially provided work when mastered is not unduly laboured. The reverse is not true; superficial recapitulation of work which requires remedial analysis may be worse than useless. Generally faults in arithmetic requiring remedial treatment were due at the time of learning to a child's poverty of experience, concrete or numerical, or to his failure to draw from his experience the underlying numerical ideas. Such attempts to learn abstract arithmetic for which he was unready frequently engender the faults in attitude to learning briefly referred to in 2·01. Undetected gaps in knowledge may also produce unfavourable attitudes to learning: a child of good intelligence may get into the habit of accepting things on trust just because what he has missed prevents him from proper participation in work for which he would otherwise have been ready.

### 4·1. The Twofold Nature of Remedial Treatment.

Remedial treatment involves a twofold analysis. The teacher must analyse the mathematical subject matter so as to appreciate its content and its stages, but it is also essential that he should find out the individual pupil's sources of difficulty. By the age of eleven individual difficulties may be very complex.

We place the question of attitude to learning first because of its importance. Faults of attitude generally accompany other difficulties, the detailed treatment of which may be part of their remedy.

### 4·2. Attitude to Mathematics.

(*a*) The most common unfavourable attitude is discouragement due to the pupil's having failed in the past to keep pace with his fellows or with what was expected of him. Freshness of approach with detailed and careful re-teaching in new contexts is the cure. So far as possible re-teaching must be disguised as something new until confidence is restored. Work must be simple enough to ensure success and the first aim must be the building of confidence through repeated successes.

(*b*) Indifference to learning Mathematics by pupils of adequate ability is not uncommon and is a difficult attitude to combat. It may occur among really intelligent pupils who react against the learning of mathematical procedures which appear to have little connection with their lives or current interests, but which almost solely make demands on care and attention. It also arises from instruction which offers too little challenge, practical or theoretical, to able and active minds. An important general question of procedure arises; of the two methods of approach (i) the preparation of mathematical techniques before tackling problems and (ii) the study of techniques arising from the needs of a problem, the former must be held responsible for much indifference to Mathematics among young people and the latter though perhaps appearing less efficient is the better for most children.

Freshness of approach and the association of what is taught with practical or intellectual interests are necessary ingredients of the remedial treatment of indifference to learning Mathematics. For instance, abler pupils with an interest in the history of everyday things may have their interest caught by a historical approach to number and measures and even to computation. Since historical development is essentially a practical one in touch with the life of earlier times such a remedial approach might satisfy both practical and intellectual interests. When co-operation has been secured it must be held by making the work not only interesting but also satisfying to the pupils' sense of achievement. Girls are perhaps more prone to this critical indifference to Mathematics than boys; this may be because many boys, looking ahead to certain desirable occupations, accept in a vague way, as a step to some distant goal, achievement without interest in what is achieved.

Many of the approaches suggested in chapters 3 and 7 could be used to combat indifference due to the feeling that Mathematics is remote from living interests.

(*c*) Another bad attitude to learning, similar in origin to indifference but common among children of lesser ability, is engendered when children are instructed in rules and become so accustomed to being shown how to perform each step that they lose all initiative and any expectation that their work in arithmetic will have meaning for them. The enjoyment of mechanical proficiency, a good thing in itself, may make this over-dependent attitude very hard to combat. Understanding has to be built up without interfering with established routines. Many of the remedial exercises in 4·3 might for instance be used to get to the root of a too superficial acceptance of how we write numbers and encourage a constructive attitude to the use of numbers in ordinary experience.

In general the most effective mode of treatment is an experimental approach to all unfamiliar situations. For instance in mensuration, which lends itself to experimental treatment, rules should at first be avoided. Pupils might even find themselves approaching afresh the notion of multiplication in finding the number of square units in a rectangle and they might rediscover the partial products of long multiplication by being confronted with a rectangle whose sides were long enough to bring the calculation outside the tables' range. For example in finding the number of square inches in the area of a pane of glass in the window a pupil might have 34 sq. in. in each of 16 rows. In looking for a way of finding the total he may see 10 rows of 34 sq. in. (making 340) and 6 rows of 34 sq. in. (making 204) making altogether 544 sq. in. This idea of rediscovery is developed further in 4·812 and also referred to in 4·5. The practical topics of chapter 3 are full of opportunities for remedial treatment for those who are over-dependent on the teacher's instructions.

### 4·3. Numbers and How They are Written.

It may be necessary to do remedial work in counting or in how we write numbers in tens and it is possible to make these a worthy activity for the eleven year old by a careful choice of context. Motive may easily be found for the making of inventories or the amassing of statistics based on a count. The statistical technique of grouping strokes in fives 𝍸, known as the gate method, is quickly adapted to counts in tens. Probably some token method of representing numbers in tens and units is worth introducing for very backward pupils, e.g. spills and bundles of 10 spills. Dull pupils may need to use counting aids for a long time, but such work is part of their normal course rather than remedial treatment.

The reader is referred to the Association's *Mathematics in Primary Schools* for detailed consideration of the elementary ideas of number, which may still be unformed in the minds of the most backward pupils. Some further suggestions are given in 6·21.

Remedial work on notation is given below in some detail, both as an example of the kind of detail that may be necessary in remedial treatment and also because a superficial acceptance of numbers written in units, tens and hundreds is often at the root of much failure to understand what is taught at early ages.

### 4·31. Remedial work on notation.

(*a*) *as denoting order*.

1. Cards with motor car numbers and letters on them to be sorted first by letters, and then into numerical order within each group.

   What keeping a register of the cars would involve e.g. GL has numbers from 1 to 9999 so at 50 lines to a page would require 200 pages.

   AGL has numbers from 1 to 999 and would require 20 pages. Boys might perhaps write out the numbers of one page—co-operatively a class might prepare a whole register: if they could organise this themselves they would have a good knowledge of order, but many would require help.

2. Numbers associated with the pupils themselves. Number on register gives class order; the school admission register will give a bigger number and a different order; the medical card numbers may be different again; each may have a 'bus pass with a number, a savings book or card with a number. Parents have voting numbers, registration numbers in the local 'co-op', works' numbers; brothers in the services have numbers which must be used as part of their address. Other numbers can be devised; ages in months, weights in pounds, heights in inches can be found for the class, e.g. in an 11-year old class ages in months will be numbers between 132 and 144. These can be arranged in order. How old will each be a year hence? or six months hence? or a year ago? Let them be found separately, and then arranged in order.

   Pupils will be interested to know that the digits of some of these numbers, e.g. service numbers and sometimes telephone numbers, have special significance, and the numbers as a whole do not denote the order of enlistment or of issue; for example a telephone number 63425, beginning with the digit 6, may indicate a city district in which all numbers begin with 6.
3. What is missing? Give say 20 numbers lying between 231 and 260 but in haphazard order. The pupil must first arrange them in order and then write down the numbers which are missing. If the numbers are on separate cards and can be moved about this exercise will be enjoyed.
4. Where does it fit in? Give a skeleton 'scale', e.g. 1900, 1910, 1920, 1930, 1940, 1950, 1960 spaced out on a long line, and get pupils to place certain intermediate numbers correctly. Alternatively the key numbers can be the tops of columns and the intermediate numbers placed in the correct columns.
5. How does it go on? Extending a series like 350, 355, 360, 365, etc., or 470, 490, 510, 530, etc.

(*b*) *emphasis on place value.*

1. Take three digits, e.g. 4, 6, 8 and consider the numbers in which they occur:

| | | | | | |
|---|---|---|---|---|---|
| 4 | 44 | 64 | 84 | 444 | |
| 6 | 46 | 66 | 86 | 446 | |
| 8 | 48 | 68 | 88 | 448 | etc., |
| | | | | 464 | |
| | | | | 466 | |
| | | | | 468 | |
| | | | | 484 | |
| | | | | 486 | |
| | | | | 488 | |

   Let the pupil read and write such numbers, in an orderly fashion, but of course not necessarily in the arrangement shown. 468, 486, 648, 684, 846, 864 without any repeated figures might be preferred for a beginning.

2. Study numbers like 32; 320; 3,200; 32,000.
Read them, write them in columns and then add them.
Halve each, read these; write them in columns and add them.
Double the answer (as a check).
Double each of the original series, read these; write in columns and add. Halve the answer (as a check).

3. Take numbers each 1 less than those just used and repeat the exercise on doubling.

| | |
|---|---|
| 31 | 62 |
| 319 | 638 |
| 3199 | 6398 |
| 31999 | 63998 |
| 35548 | 71096 |

Incidentally note that this uses *all* the 10 digits.

(*c*) *emphasis on pattern.*

Many secondary school entrants may have been assumed to have noticed the recurrent pattern of the natural numbers, when in fact they have not. Very simple exercises can be given.

1. Writing from dictation at first, and finishing each column as they find they can.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 10 | 100 | 1000 | 11 | 101 | 91 |
| 2 | 20 | 200 | 2000 | 22 | 112 | 82 |
| 3 | 30 | 300 | 3000 | 33 | 123 | 73 |
| 4 | 40 | 400 | 4000 | 44 | 134 | 64 |
| 5 | 50 | 500 | 5000 | 55 | 145 | 55 |
| 6 | 60 | 600 | 6000 | 66 | 156 | 46 |
| 7 | 70 | 700 | 7000 | 77 | 167 | 37 |
| 8 | 80 | 800 | 8000 | 88 | 178 | 28 |
| 9 | 90 | 900 | 9000 | 99 | 189 | 19 |

Other exercises adding 11s or subtracting 9s can be devised.

2.

| | | | |
|---|---|---|---|
| 9 | 90 | 99 | 180 |
| 18 | 81 | 108 | 171 |
| 27 | 72 | 117 | 162 |
| 36 | 63 | 126 | 153 |
| 45 | 54 | 135 | 144 |

3.

| | | | |
|---|---|---|---|
| 2 | 12 | 22 | 32 |
| 4 | 24 | 44 | 64 |
| 6 | 36 | 66 | 96 |
| 8 | 48 | 88 | 128 |
| 10 | 60 | 110 | 160 |

(*d*) *emphasis on calculating.*

1. Read these numbers: 3; 33; 333; 3333. Then write their doubles.

Repeat with 30; 300; 3000. Then with 42; 420; 4200; 23; 230; 2300 etc., N.B. no carrying involved.
Use numbers like 8; 80; 800; 8000; 84; 840; 8400, etc. Read, and then halve.
Then numbers like 24; 240; 2400. 36; 360; 3600, etc.
Progress to 240; 242; 244; 246; 248, noting the pattern of the even numbers.

2. Similarly work with multiplication and division by 3 using numbers which would not probably have been grouped together in the Primary School where the process is likely to be learnt first with all numbers under 100, and the extension of the notation has not been sufficiently considered.

In all this work it would be helpful to have a supply of printed numerals each on its own card, say a $\frac{3}{4}$ inch square, so that pupils could make numbers by moving the numerals about without having to write. Letters for word-making are common, but figures for number-making are not usual.

Number cards for 1, 2, 3, 4, etc., 10, 20, 30, etc., 100, 200, 300, etc., are however obtainable.

3. Drill can be made interesting by the use of recurrence and sequence.

(i) $17 \times 3$
$27 \times 3$
$37 \times 3$
$47 \times 3$
$57 \times 3$
$67 \times 3$
$77 \times 3$
$87 \times 3$
$97 \times 3$

Before the work starts make sure by oral questioning that the pupil

(i) has read, and not merely copies *all* these numbers, and that he can read them out in order.
(ii) has noticed the recurrence of 7.
(iii) knows $7 \times 3 = 21$.

(ii) $11 \times 4$
$21 \times 4$
$31 \times 4$
$41 \times 4$
$51 \times 4$
$61 \times 4$
$71 \times 4$
$81 \times 4$
$91 \times 4$

Oral questioning both before and after the working of this sequence would be wise. It should include emphasis on the fact that all the products are *even* numbers and can be exactly divided both by 2 and 4.

(iii) A further sequence can be set, or arranged as two sequences $x$ and $y$.

$98 \div 2 \ldots\ldots\ldots x$
$88 \div 2 \ldots\ldots\ldots\ldots\ldots\ldots y$
$78 \div 2 \ldots\ldots\ldots x$
$68 \div 2 \ldots\ldots\ldots\ldots\ldots\ldots y$
$58 \div 2 \ldots\ldots\ldots x$
$48 \div 2 \ldots\ldots\ldots\ldots\ldots\ldots y$
$38 \div 2 \ldots\ldots\ldots x$
$28 \div 2 \ldots\ldots\ldots\ldots\ldots\ldots y$
$18 \div 2 \ldots\ldots\ldots x$

(iv) $11 \times 9 = 99$
$22 \times 9 = 198$
$33 \times 9 = 297$
$44 \times 9 = 396$
$55 \times 9 = 495$
$66 \times 9 = 594$
$77 \times 9 = 693$
$88 \times 9 = 792$
$99 \times 9 = 891$

The sequence $99 \times 1$, $99 \times 2$, $99 \times 3$, $99 \times 4$ etc. will of course exhibit the same set of results.

Pupils doing this should be watched. Some will notice the sequences part way down, and will write the rest of the answers without doing the multiplying. Others will not notice any sequence; these should be helped to see it at the end as a check, particularly if they have one wrong.

(*e*) *emphasis on meaning and use.*

1. For miscellaneous practice numbers can be found in, say, a newspaper and read out for pupils to write down from dictation. On a single page of a daily newspaper the following were found:

11 pits
44,600 men idle
labour force of 140,000
115 pits
3 pits
2600 men
13 days
272,000 tons
75,000 tons
10 Downing Street
79 routes
5 per cent
500 copies
£20
3000 branches
725 branches
60 per cent.
25 per cent.
2 socialists
aged 82

4000 busmen
12 garages
23 routes
17 routes
793 buses
3645 men
293 routes
700 crews
1300 crews
600 workers
£960 million
509th group
393rd squadron
25 ft.
100 ft.
150 ft.
7487 men
1471 buses
100 murders
435 surrenders

2. Reading, and recording, numbers by means of 'isotypes' (6·217) leads to counting in fives, tens, twenty-fives or hundreds, etc., and these numbers can then be expressed in the usual notation. Thus, among the numbers above, 3000 branches and 725 branches might be represented by a diagrammatic shop-window for each 100 branches, or the numbers of 'buses by a pictured 'bus for every 50, and the crews by a little man for every 20 crews.

### 4·4. Understanding the Four Rules and Learning Tables.

Most pupils of eleven are still unready for a fully reasoned approach to rules, and therefore remedial work is best approached by making sure that the foundations are firm. The elements of these foundations are treated in detail in the paragraphs which follow; some have already been enumerated in 2·41 in connection with diagnosis.

**4·41.** A common complaint about a pupil's knowledge of arithmetic is that he 'does not know his tables'. This complaint is generally meant to imply that he has failed in a simple act of memorisation, but in fact it more often covers deep-seated failures to learn about numbers from experience for which drill in results is no cure.

In the first place the addition–subtraction and multiplication–division tables are not learned usefully unless these operations and their interrelationships have meaning for the pupil. To help pupils to gain this understanding is the most difficult and delicate task of the teacher of arithmetic. It is dealt with at some length in the Association's *Mathematics in Primary Schools*. Evidence that it has been performed with some success is afforded by a pupil who uses a higher level skill for some practical purpose. For example finding the cost of 7 things at 5 pence each might be performed at various levels. The most primitive first stage would be to lay out 7 rows of 5 pennies, or counters, and count up the total one by one. At a higher level the reckoning might be by addition $5+5+5+5+5+5+5$. The most efficient way is by multiplication, $7 \times 5 = 35$. Dealing with 35 pence might similarly reveal stages of efficiency from a crude piling of pennies in twelves to a recognition that the result was 1 penny short of 3 shillings.

Remedial treatment consists in the careful experimental approach to every new context. There may be a great deal of latent knowledge to which new experiences can be related provided that the pupil produces as a basis some way of working in which he feels secure. Thus in the simple example given above, the pupil who produces the addition of fives may be encouraged to say that he 'has added seven fives' and by his form of words connect the result with a half remembered 'multiplication table'. The use of counters or diagrams may help him both to see and to describe in words what he has to do.

The change from one unit to another in using money and measures is an important example of the use of equal groupings, e.g. 3 feet to a yard, 8 pints to the gallon, 8 half-crowns to £1. Because of the concrete nature of the problem this unit changing is a valuable example of the use of multiplication and division tables, which can be built if necessary upon the clumsier methods of repeated addition and subtraction.

**4·42.** The actual memorisation of number bonds owes much to experience of using numbers in everyday life, for example the parts of 12 are known from experiences involving 12 pence in 1 shilling, adding sevens may be associated with finding the date a week hence and so on. In 6·2 and in the Association's *Mathematics in Primary Schools* the contribution of pattern to number knowledge has been pointed out, e.g. in the latter (3·22) the table patterns on the number square showing in order the numbers from one to a hundred. There is in fact a wide diversity of ways of approaching table-learning and it is only in the final stages when there is already a good basis of remembered number facts that 'sharp practice' in answering questions on number bonds is helpful. Indiscriminate use of sharp practice may shake confidence and hinder understanding.

The extreme case of tables failure, that of counting in ones, perhaps on the fingers, for the addition–subtraction bonds within 20 is not uncommon even among abler pupils. It is generally a bad habit acquired when written calculations are begun very early before the child has enough experience of using small numbers or enough encouragement to show he knows results from his previous experience. Few pupils of eleven are in doubt about what is meant by addition so that the addition aspect of the table can be readily dealt with and the finger-counting habit eradicated by selecting a few known results and working from these. Rhythmic counting in twos, threes etc. may also form a useful framework into which to fit sums of two unlike numbers. The items of the subtraction table and the meaning of the operation should be associated with the corresponding additions. It is probable that subtraction is less well remembered and not so clearly understood as addition. In mental work it is generally performed as a complementary addition.

**4·43.** Even those who know the addition table to 20 may require help in extending it from items such as $8+6$ to $18+6$, $28+6$ etc. Here the pattern of the series should be noted

$$\begin{aligned} 8+6&=14\\ 18+6&=24\\ 28+6&=34 \text{ etc.}\end{aligned}$$

with the corresponding subtractions.

$$\begin{aligned} 14-\ \ 8&=6\\ 24-18&=6\\ 34-28&=6\end{aligned}$$

These extended skills are called upon in written computation.

**4·44.** The multiplication–division tables are related to the series formed by repeated additions. It is a step on the way to knowing that $8\times 4=32$ to know that 32 occurs in repeated counting of eights.

Particular items may be known from usage. Thus $12\times 3=36$ may be known from the number of pence in 3 shillings or of inches in 1 yard.

Knowledge of particular numbers may contribute items to a table. Thus the factor structure of 60 may be known in various forms from work connected with the clockface and 60 minutes to the hour.

**4·45.** For use in written computation it is important to be at home not only with the multiplication–division tables themselves but with extensions of them. For example $3\times 7$ must be known and also $30\times 7$, $3\times 70$, $30\times 70$ etc. If a pupil has reached the stage of a ready flexibility in the use of factors in a continued product these will present no difficulty; he will say, e.g. $30\times 30=3\times 10\times 3\times 10=9\times 100$. A slower learner may proceed by examining the pattern of a series of results found by more laborious methods. For example in writing down the number of shillings in £1, £2, £3, £4, etc., the results appear as 20, 40, 60, 80, etc. shillings. The relationship to the table of twos may be noted and what is true for 1, 2, 3, 4, etc. twenties may be intuitively applied to thirties, forties, etc.

**4·46.** Faulty memorisation is so frequently deplored that some points about it are here gathered together.

It may often be due to the pupil's failure to build up associations with particular numbers, either of particular numbers with everyday life or of numbers among themselves. Each number bond for example may be approached as an item no more memorable than any other except perhaps for its verbal rhythm; thus 'five fives are twenty-five' may roll more smoothy off the tongue than 'six fives are thirty'. The latter however may be better remembered by another child who knows there are six five minute intervals on the clock face for half an hour. Many examples of helpful association could be given.

It is a help to have noted that $6 \times 5 = 5 \times 6$, e.g. when counting milk bottles in a crate.

Some knowledge of number characteristics, e.g. odds, evens, primes, etc. and of sequences of numbers, e.g. the pattern of counting in nines, see 4·31 (*c*) 2, may also help memorisation.

Facts about measures are clearly best remembered by association with actual measuring. Since units of measurement are of arbitrary choice a cure for faulty memorisation may sometimes be found by considering the history of the choice of unit and of its naming.

Faulty memorisation of procedures such as the rules of written arithmetic, arises mainly from the putting together of a number of elements which have been imperfectly assimilated. If numbers too large for mental reckoning occur when the rules for dealing with them are still unsure, an experimental approach may both deal with the particular situation and help to remedy the rule. Apparatus for representing numbers in tens and units should be available for those who require it. It will be specially helpful in showing that results can be obtained by dealing separately with the tens and the units.

### 4·5. Written Work in the Four Rules of Number.

The last paragraph deals with faulty memorisation of rules generally due to a faulty approach. Although ways of computing should become habits of working they should fundamentally be built up from known elements, so far as possible experimentally. An example of this sort of thing is given in 4·2 (*c*). This shows clearly how in finding a number of square inches in a window pane the multiplication of 34 by 16, which are numbers just outside the range of working without the aid of paper and pencil, is tackled by dealing with partial products, first 34 by 10 then 34 by 6. That this is allowable depends in the pupil's mind on the particular example of squares in the window pane. It is most important therefore that an experimental approach to rules should have a real setting or at least that the numbers should be represented by some concrete material such as beads or spills with convenient bundles of ten units.

The other elements going to make up the rule, i.e. elements additional to this question of dealing with the numbers piecemeal, have been dealt with in paragraphs 4·3 and 4·4.

In 4·812 remedial reckoning is shown in connection with finding the area of rectangles.

For some children entering a Modern School some of the rules may be new work. Such rules can be approached experimentally at a simple number level arising out of some interesting context. The only bar likely to hinder progress is one of attitude as described in 4·2 (c). Pupils may be so accustomed to being shown what to do that they cannot discover even a clumsy method of working for themselves. Any clumsy method, even counting in ones, can be used as a foundation for something better. Sometimes a method which is not a general one may show correct thought; for instance the pupil who has to divide 43 by 3 may take 36 and 7, obtaining (12 + 2) and one over, instead of dealing first with 4 tens in the conventional way.

### 4·6. Money and Measures.

Money and measures are often grouped together in primary arithmetic, and are therefore so grouped in this remedial chapter.

**4·61. Money.** The approach to reckoning with money suggested in 6·7 is readily adapted to the needs of an entrant of any ability or attainment. The most likely trouble calling for remedial treatment is confusion and discouragement due to learning ways of reckoning with money as though they were a number of arbitrary new rules.

In treating this state of mind it is wise to leave rules alone for a time and to concentrate on mental computation with small amounts or on computation with round sums of money. The build-up from this to some form of written reckoning is described in 6·7, which includes the construction of ready reckoners. For the very backward concrete aids may be needed even at the secondary stage.

Calculations with large or awkward amounts of money can well wait until some practical situation makes them necessary. In any case all reckoning with money should be kept closely in touch with the needs of everyday life.

**4·62. Measures.** At the entrant stage the most likely failing is a lack of practical acquaintance with measures. This is a contributory cause of failure to memorise tables of measures.

A lack of knowledge of the ways of setting down calculations with measures need not be considered as a gap requiring remedial treatment (6·83).

A historical approach to our common units of measurement often proves a refreshing remedial revision.

### 4·7. The Part played by Language.

One test of whether work undertaken is remedially effective is the growth of the pupil's powers of talking about what he is doing. The pupil who can describe his working, however clumsy, has confidence in its reasonableness; if he can say what he is finding difficult he is a long way towards solution and his teacher has a clue to what is needed.

Pupils who have a mechanical knowledge of rules, acquired perhaps by much practice, often fail to apply rules in 'problem' situations. The problem set verbally is a particular form of problem much prized because of its appearance in examinations. Problems set as a practical situation, i.e. something to be done, perhaps quantities to be measured and paid for, lend themselves very well to verbal description, for example after measuring a window for curtains,

> I measured the width of the window, allowed a third for fullness, chose material of a given width and decided I must have a width and a half in each curtain, that is three widths for the pair etc.

This is not only a more useful exercise than solving a problem with given measures set verbally, but it is also a preparation for understanding verbal instructions about curtain making.

Remedial work on problems of whatever type must include opportunities for verbal description of work accomplished.

A pupil may find his solution of a problem involving small numbers hard to explain either because it has been found ' in a flash ' or because the circumstances have been visualised and a practical rather than an arithmetical solution obtained. Problems with larger numbers however must be analysed arithmetically before being solved; they are therefore harder to solve, but the steps of a solution obtained are more easily recognised and described.

### 4·8. Geometrical Knowledge and Area-finding.

The work considered as needing remedial treatment has been mainly numerical. Other mathematical opportunities in the Primary School seldom pass beyond the initial stages so that though individuals may have gaps in their knowledge there are seldom deep rooted misunderstandings.

Occasionally words, e.g. 'straight' or 'triangle', may be habitually misused but correction is merely a matter of establishing a better informed habit.

**4·81.** Measurement of area is however a geometrical subject often broached in Primary Schools. It is not uncommon to find the preliminary geometrical stages hardly treated at all, and in these circumstances the rule for calculating the area of a rectangle may be used without understanding and even inappropriately for other shapes.

The remedial treatment of such a condition has three main stages:

(1) A clear notion must be established of what is meant by area.
(2) The need for a unit must be established and it must be chosen in size and shape.
(3) The basic process of square counting must be re-established before deducing rules for particular shapes.

If the idea of ' perimeter ' has been introduced at much the same time as that of ' area ', the resulting confusion may call for very careful treatment of stage (1). It is a mistake to associate in teaching two such new terms which have nothing in common.

**4·811.** Two examples of approaches to area-finding satisfying some of these conditions are now given and they are carried on into the numerical stages to show what valuable remedial number work can be associated with area calculations at any level of number difficulty.

*Example* 1. The practical problem might be posed of comparing the size (surface) of two leaves of different shape. This is a problem of practical significance since leaf surface area is of importance to the plant.

Comparison between the two leaves could be made in the following ways:

(*a*) Place one leaf upon the other. If one does not lie wholly within the other, the possibilities of dissection might be considered, pieces being matched until some of one leaf remains.

(*b*) Draw patterns of the leaves on paper and try the above dissection without destroying the specimens.

(*c*) Draw patterns of the leaf on squared paper (a natural suggestion if the work is remedial) and find a measure of the size of each leaf by counting squares and part squares. The squares can be counted first in rows and then as a check in columns.

The ways of reckoning part squares can be discussed; counting parts more and less than a half can be tried.

The leaf can be redrawn so as to fall differently on the lattice of squares and a new assessment made.

The advantage of using an irregular shape is that it emphasises (3) above, the fundamental technique of square counting; (2) above, the choice of a unit, has been taken for granted but supplementary work could be added from the suggestions in 6·5. A note on the use of squared paper is included in 4·812.

*Example* 2. A question of size might arise in discussion of the school hall and how it compares with a classroom. Floorspace rather than airspace would be likely to be considered first. Seating capacity might suggest itself as a first measure, but the hall would have a seating capacity on chairs and the natural unit for the classroom would be seating with desks. Standing room might also be considered.

The next question might be 'How many classrooms could be partitioned off in the hall?' Here shape is important and area calculation does not serve. Suppose the hall to be 70 ft. by 30 ft. and the classroom 24 ft. by 22 ft. The classroom must be mapped out in the hall by marking off three of the shorter sides along the longer side of the hall, for $3 \times 24 = 72$, i.e. more than 70. A discussion might follow as to whether the pieces of the hall left over though different in shape were as much as one classroom. If scale drawings, say 10 feet to the inch on tenth inch paper, are used, numerical assessment by square counting could be made. Used as a remedial exercise for 4·81 (1) and (3) above the enquiry would be complete.

**4·812.** Such an enquiry could however be used as a remedial exercise in multiplication at various number levels as indicated below, if the areas of other rectangles are then examined.

Each item of the following is supposed to be accompanied by a diagram and additional examples of the type to be taken according to the need of the pupil. The work can be terminated at any appropriate stage. Convenient units are supposed. Squared paper with $\frac{1}{4}$ inch or $\frac{1}{8}$ inch rulings is to be preferred to finer rulings for square counting, but when counting in tens supersedes counting in ones inches divided into tenths can be used for diagrams, especially if squares are lightly marked or shaded as they are reckoned.

| *Rectangle* | *Number work* |
|---|---|
| (*a*) 7 by 4 | count by rows and count by columns.<br>revision and meaning of multiplication table also $7 \times 4 = 4 \times 7$ $(a \times b = b \times a)$. |
| (*b*) $7\frac{1}{2}$ by 4 | note $7 \times 4$ and 4 halves, continued at (*i*) below. |
| (*c*) 20 by 6 | by counting and addition.<br>finally expressed as multiplication, $20 \times 6$.<br>revision of a first extension of the multiplication table. |
| (*d*) 20 by 20 | as above extended table, second stage. |
| (*e*) 24 by 6 | diagram showing partial products $20 \times 6$ and $4 \times 6$.<br>revision of first stage of written multiplication. |
| (*f*) 20 by 25 | easy special case of (*g*). |
| (*g*) 20 by 24 | first stage long multiplication. |
| (*h*) 22 by 24 | diagram showing two rectangle corresponding to the partial products $24 \times 20$ and $24 \times 2$ (or $22 \times 20$ and $22 \times 4$) (Diagram 4). |

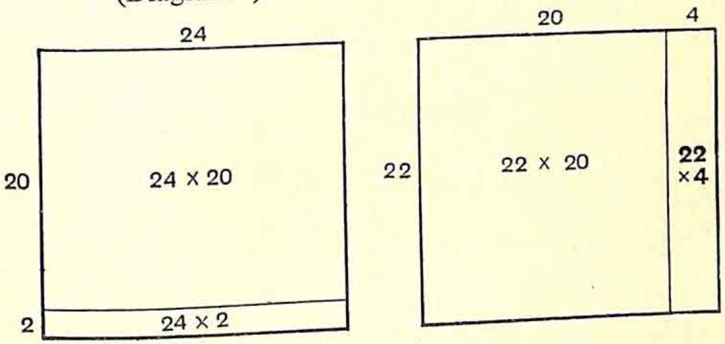


DIAGRAM 4.

*Fractions*

| | |
|---|---|
| (*i*) $7\frac{1}{4}$ by 3 | fraction 'multiplied' by an integer, or see (*b*) above. |
| (*j*) $\frac{1}{2}$ by $\frac{1}{2}$<br>$\frac{1}{2}$ by $\frac{3}{4}$ | fraction 'multiplied' by a fraction, diagrams by pupils carefully interpreted and other harder examples (Diagram 5). |
| (*k*) $\frac{1}{2}$ by $1\frac{1}{4}$<br>$2\frac{1}{2}$ by $1\frac{1}{2}$ | examples such as these will in the first place be dealt with 'partially' and not as improper fractions thus $\frac{1}{2}$ of $1\frac{1}{4}$ gives $\frac{1}{2}$ and $\frac{1}{2} \times \frac{1}{4}$ and the diagram for $2\frac{1}{2}$ by $1\frac{1}{2}$ is in four parts 2 by 1, 2 by $\frac{1}{2}$, 1 by $\frac{1}{2}$ and $\frac{1}{2}$ by $\frac{1}{2}$. |
| etc. | including decimal fractions. |

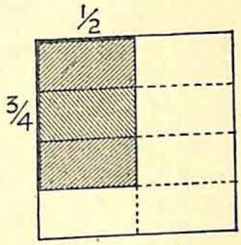


DIAGRAM 5.

## 4·9. Gaps in Knowledge, Remedial Work Later in the Course.

Gaps in numerical knowledge have been dealt with in a general way only. A warning may, however, be appropriate. Once such a gap has been made good the remedial treatment of later stages may be accomplished more rapidly than with pupils whose remedial needs arise from their general unreadiness at their first learning. With the latter care must be taken to give specifically remedial work a little at a time, as all work involving understanding requires periods for assimilation in which there is no active teaching.

**4·91.** It has been assumed so far that difficulties requiring remedial treatment have arisen at the primary stage. There may however be later occasions when there is need for remedy, perhaps because a pupil has been frequently absent in the early years in the secondary school, or has entered the school later after a different or unsuitable course elsewhere. Again a whole class may suffer through unavoidable changes of teacher, or through beginning with an unskilled member of staff.

The principles stated in 4·1 will also apply in dealing with later difficulties and gaps in knowledge. Attitude to learning is the first concern. Confidence must be re-established and initiative encouraged by work which is fresh in context, and simple enough to ensure success.

Discussion of the Mathematics which is normally learned between 11 and 15 years of age is in chapter 6. Remedial treatment of this calls for appreciation of essentials on the part of the teacher, and for new ways of approaching work which pupils have found confusing. Such pupils often have difficulties because they have been taught procedures with notations which are so unfamiliar that the procedures, and even the notations themselves, seem to have no meaning or usefulness. For example, algebra learned before the use of letters for numbers has become a familiar practice may involve attempts to apply memorised rules unrelated to fundamentals of number; and the same sort of difficulty may occur with the decimal fraction notation and rules learnt for computing with it.

Although the greater experience and maturity of older pupils may make re-teaching easier they will not willingly undertake the practice which ensures mastery of techniques unless such mastery is shown to be useful and worthwhile.

## CHAPTER 5

# ORGANISATION WITHIN THE SCHOOL

### 5·0. Differences between Schools and the Range of Ability of their Pupils.

We suggest in chapter 1 that the boys and girls with whom this report is concerned are to be found in every type of Secondary School, but the majority are in schools, or sections of schools, which are organised separately for them and in which it is therefore possible to make provision for the mathematics teaching unhampered by the claims of pupils who may later specialise in the subject. Even so the range of ability and of interest in Mathematics amongst the boys and girls that we have for convenience referred to as modern school pupils is very great, and the circumstances in which they are taught, even if we omit the special problems of the Comprehensive School, are almost bewilderingly varied. Schools may vary in size from one to six or more classes per age group; they may be for boys or girls separately or mixed schools; they may be in ample buildings or in old and crowded premises. Again they may be rural or urban or draw pupils from both town and country; they may be suburban with pupils who look forward to work of a clerical or business type, or serve a neighbourhood where the parents are in skilled artisan employment expecting their children to follow them in a skilled trade; the neighbourhood may be scattered; or the school background may be dull and colourless and the parents unambitious for their children. The aims and organisation of schools will in short differ so greatly from one to another that advice must be confined to general considerations and fitted by the individual school to its circumstances.

### 5·1. Staffing.

A determining factor in what Mathematics is undertaken is the staffing of the school. Some schools, generally those where Mathematics is looked on as mere reckoning, beg the question of staff organisation by making Mathematics a form subject, so that all form teachers are involved in teaching it. For the least able pupils form teaching of a number of subjects including Mathematics is often best. One reason for this is that these pupils generally still lack some of the fundamental ideas of number which are best learned incidentally as they arise. In general however unless the school is unusually favoured in its staff from the mathematical point of view, there will be several form teachers who are not interested in Mathematics or qualified to teach it in a way which will refresh the arithmetic of the Primary School and extend their pupils' ways of thinking mathematically. There should in fact in every secondary school organisation be means by which some teachers can avoid the teaching of Mathematics. This means that there must also be on the staff one or more teachers prepared to take more than one class for Mathematics.

It will be advantageous if amongst the teachers of Mathematics there is one able to act as leader of the team and to be responsible under the Head for co-ordination of the work of colleagues and for development of the syllabus. Occasionally there may be someone whose qualifications are such that he or she teaches no subject other than Mathematics, but more modified forms of specialisation are from many points of view advantageous to mathematics teaching as well as desirable in other ways.

A favourable arrangement might be to have from two or three up to about half the staff teaching Mathematics and for several of these teachers to be concerned with teaching more than one class and for most of them to teach at least one subject other than Mathematics. Certain combinations of subjects are specially favourable, for instance at some points in the course it is very convenient if a teacher is concerned with both Mathematics and Geography in a class. The team leader should be prepared to keep in touch with new ideas about the teaching of his subject. There should be arrangements for periodical meetings of the team for discussion.

It should not be assumed that the teaching of traditional arithmetic is all that can be expected of members of staff with meagre mathematical qualifications. Practical topics are often best tackled by such teachers and provided there is advice available, so that mathematical implications of a topic are not overlooked, sounder progress may result than from attempts to follow the Pure Mathematics of a text-book. Indeed practical topics call for skilled class management, but often for simple Mathematics only.

If a high degree of specialised teaching is adopted, co-operation between mathematics teachers and other members of the staff is of very great importance. The occasional project in which children follow up some special interest in ways which cut across subject boundaries can be more easily dealt with in partially specialised organisations.

Reorganisation of forms into sets according to mathematical ability is sometimes arranged if staffing conditions are favourable. The pros and cons of this arrangement are discussed in 5·2.

Every school has to weigh up its own personal problems in this question of allocation of staff. The main objectives mathematically should be adequate co-ordination of effort and guidance from someone with the time and the insight to make Mathematics both a true means of education and a useful tool. Entrant classes require teachers who have insight into the children's difficulties, and originality enough to make a refreshing beginning to work in a new school. The oldest pupils on the other hand are reaching a maturity which makes possible ways of working closed to them at earlier ages. The extension of their range of mathematical work has to go on side by side with work in which the extension of their social interests demands further application of what they have already mastered especially in arithmetic. This dual opportunity requires a versatile teacher.

### 5·2. Sets for Mathematics.

The term *set* is used to denote a teaching unit drawn from more than one form. Setting for Mathematics involves re-classifying two or more forms for all their work in the subject, and therefore generally, though not always, involves time-tabling so that all forms concerned in the setting take Mathematics at the same time. There must also be enough mathematics teachers for the number of sets.

Very occasionally it may be possible to arrange for more sets than there are forms, thereby gaining an immediate advantage from the reduced size of teaching unit. In any other circumstances the pros and cons of setting need very careful examination before the organisation is adopted.

**5·21.** The organisation is most often used to reduce the spread of ability in the subject within the teaching units, that is as an alternative to some of the group and individual work which might otherwise be necessary within classes. Whether there is a balance of advantage in this re-classification depends mainly on

(*a*) the number of pupils transferred with advantage up or down from their normal teaching unit.
(*b*) the staff available.
(*c*) the extent of difficulties caused by limitations on time-tabling.
(*d*) the amount of disturbance through breaking up the usual teaching units.

Sets, especially for older pupils, are also used to meet differences of interest as well as of ability and, or, to give one set more time for the subject than another.

**5·22.** Partial setting, that is setting for part of the mathematical time combined with form teaching for the remaining periods, tends to break up the unity of mathematical teaching and to introduce difficulties of syllabus. Where a set drawn from two forms has extra time for Mathematics, it is important that the pupils having shorter time should have at least one period of teaching as a set, otherwise difficulties of syllabus-making are almost insurmountable. A common example, referred to in 5·5, is that in which boys of two forms are considered as a set for additional geometry; the girls' work is almost sure to suffer unless they have at least one mathematics period in which the balance of their syllabus can be dealt with separately.

**5·23.** In most organisations all the pupils of an age group who are both dull and backward, that is the least able described in 5·36, are classified together and will not be affected by setting unless there are more sets than forms. Even if a few of the lowest form have rather greater mathematical ability than their fellows, it should not be too difficult to deal with them within the class.

Remedial work for the backward but intelligent in higher forms is a temporary and individual matter; demotion to a much lower set is not a complete solution because of their much greater rate of progress.

Slower pupils benefit from the kind of integration of curriculum only possible in a form unit. Even pupils of moderate ability may lose much if opportunities for integrated studies are reduced by the breaking up of forms into sets. The question of following topical interests, stressed in chapters 2 and 3, makes setting of very doubtful advantage except for older and abler scholars. Topical studies are not impossible in sets, but contacts with other subjects are made more difficult.

## 5.3. The Syllabus.

A syllabus should be treated as a guide and not as a goal. This means that it must be flexible and adaptable to the needs of the particular pupils in charge of any teacher, and to the interests and opportunities which arise naturally during any school term.

In framing a syllabus for a four or five year course therefore, it would be wise not to lay too much stress on the order of the items included, and not to attempt to prescribe in too much detail the methods of study. Technical details which must be consistent throughout a school may be set out in a separate document.

The syllabus most typical of a school is that for a class of average ability within it. It is a mistake to take the syllabus for abler pupils and modify it to suit the average: the reverse process is wiser.

We begin by considering a minimum of everyday useful Mathematics which should be included in every syllabus.

**5·31.** The computational skills essential to everyday affairs are few and simple (6·0). Any work in the Secondary School which is specially directed towards attaining these simple skills will differ from similar work in Primary Schools in approach and application more than in mathematical content. All but the very backward will at least have broached at the primary stage most of this essential arithmetic, and the chapter on remedial work contains suggestions that can be used as remedy, as revision, or as approach.

**5·311.** The following points are important for every syllabus:

(*a*) Economy of working is often obtained by extending the range of work, e.g. computation using fractions may supersede some earlier ways of working.

(*b*) Much which is, or which appears to the pupil to be, new work does in fact provide practice of an elementary kind, e.g. a graph by representing magnitudes along axes gives both insight into number and practice in computation when expressing magnitudes to scale.

(*c*) No syllabus will be complete without provision for the development of ideas of space and some work in mensuration.

(*d*) Graphical representation should also find a place in every scheme.

(*e*) All syllabuses will have at every stage the theme of increasing the pupils' number competence and spatial knowledge, but this should be the outcome of work which has other purposes and other claims on the pupils' attention.

(*f*) No syllabus should be restrictive in the sense that it defines some target which should be attained without digression.

**5·32.** Beyond such simple considerations syllabuses will differ, not only from school to school, but also from class to class within an age group.

A syllabus can be thought of in two distinct parts—one a list of mathematical ideas and principles and skills, the other a list of the topics through which this Mathematics will be related to the pupils' lives. These two lists can be set out in parallel columns, or in alternating paragraphs, but should not be wholly separated, for the topics will provide a vehicle for learning Mathematics, as well as a field of application for what has been learned.

**A teacher should be very chary of including any particular mathematical idea which he cannot relate to some situation which is real to his pupils :** on the other hand if pupils become interested in abstract Mathematics, or in mathematical puzzles, these interests should be encouraged and not checked because they have not been placed in the syllabus.

Often several topics are available in connection with the same mathematical idea, and different ones may interest different groups of pupils within the same school. An example of setting out a scheme in parallel columns is in 3·31, but there the topic takes precedence; in making the syllabus for the school, teachers may prefer a form more like the following:

| | |
|---|---|
| Introduction of tangent of an angle | Shadow stick. |
| Further uses of tangent of an angle | Gradient of a graph; direction of track to go $x$ miles E. and $y$ miles N. etc. |
| Introduction and use of sine of an angle | Resolving a vector quantity; slope of a road, etc. |
| Use of straight line 'conversion' graph | Degrees Fahrenheit to degrees Centigrade; inches to centimetres; pints to litres; etc. |

The various teachers working with the same syllabus must co-operate as far as possible, but account should also be taken of any special abilities or experiences which they have had: some may be in a position to make navigation more interesting than surveying, or to introduce some topic which we have not mentioned. We have tried to give variety in our suggestions and to include much more than could go into any one syllabus, but we make no claim to have exhausted the possibilities.

We recommend that each teacher should make a syllabus for each of his own classes at the beginning of the school year, taking into account of course what his pupils have done in the previous year, and any matters agreed with his colleagues as to the work of the school as a whole. This syllabus can be more detailed than the four or five year syllabus, but the same principles apply—it should be flexible, and related to real topics of interest to the particular class for which it is made. **Record of what has been done and the manner of doing it become increasingly important as the course proceeds.**

**5·321.** Although there are good teachers who make little special provision for practical work, depending rather upon contacts with work in other subjects and on short excursions into matters of general interest, we nevertheless recommend that most syllabuses should actually specify practical studies which may be valuable at various stages and that many should include some continuous practical course such as those described in chapter 7.

In many schools the syllabus may go much further and specify practical matters which the mathematics teacher will be expected to deal with thoroughly as a contribution to the pupils' general knowledge as well as to the understanding of the Mathematics learned. It is quite usual to find the economics of everyday life treated in this way, but schemes might include other matters of interest and importance.

**5·322.** In shaping a syllabus special considerations are applicable at different stages. The problem of building up a course will therefore be considered in successive years, but throughout there will be the overriding need to implement the main objectives of the course.

**5·33. The first year.** Chapter 2 deals with the early exploratory period of the first year. Freshness of approach is emphasised. Formal testing is deprecated as undesirable in itself and unnecessary. Various ways of finding out what the teacher needs to know about his pupils, while interesting them in something new, are set out in detail. Emphasis is also laid on practical measuring as an extension of the work of the Primary School and perhaps as a beginning to surveying. Revision is dealt with only as it arises.

It is recommended that this kind of approach should be maintained throughout the year, but with an increasing amount of work which is new mathematically. Special attention is directed to the general paragraph 5·31 above and also in particular to 5·311 (*b*), (*c*) and (*d*).

The programme of spatial work should be carefully thought out. Some of the early work is observational and linguistic; it may be almost incidental. Boys may have had more opportunities than girls for using geometrical instruments; consideration should be given to the needs of both. Skill in draughtsmanship may be acquired incidentally in practical work within or outside the mathematical course.

The main differences between classes will be in their readiness to enlarge the field of number work. Mensuration will often give a clue to the pupil's stage in number. In particular readiness for vulgar and for decimal fractions will vary greatly, but some work involving the use of fractional notations in practical contexts should be included in all classes (6·151). It is unlikely that even the most advanced pupils are completely at home with the notations at this stage.

A separate paragraph, 5·331, is devoted to fractions since the work may fall in the first, the second or later years. The placing of vulgar and decimal fractions should be considered separately.

**5·331.** The notation of vulgar fractions is for the majority of pupils established in simple practical contexts at the primary stage, but the learning of new measures, such as degrees of angle, introduces opportunities for revision and extension. Equivalence of fractions can also be treated early in the course. The four rules for vulgar fractions may have been learned at the primary stage by rather abler pupils, but, if so, the rules have probably not been fully established in use and an experimental approach in a new context as suggested in 6·246 is valuable for all and may bring together those who have and who have not learned the rules before. In some classes computation with vulgar fractions may not be attempted until the second year when a decision must be made whether pupils are likely to use fractional methods of computation freely or to rely on direct methods only.

Whenever fractional methods are learned, or revised for the first time, they should be introduced in some practical context and established firmly but without elaboration.

The decimal fraction notation is less freely introduced, and less adequately dealt with when introduced, at the primary stage. In most first year classes work involving construction and measurement using tenths, and perhaps estimated tenths of tenth, of an inch is valuable whether the notation is new to the pupils or not. When metric measures are learned, generally in the first or second year, they should be introduced practically and used as a means of extending and establishing understanding of the decimal notation. This calls for close co-operation between mathematics and science teachers to avoid duplication of effort or misunderstandings. The fundamental experiences of metric measures may be given either in the science laboratory or the mathematics room, but the extension of mathematical expression to decimals should accompany them. It is not satisfactory to learn metric tables and decimals theoretically before practical treatment, nor is it sensible to postpone proper statements about metric measures because pupils are not at home with decimals. Many pupils require considerable time to become accustomed to decimal fractions before undertaking any but the simplest calculations with them.

The four rules for decimals may sometimes have been attempted at the primary stage, but they are often only learned as rules of thumb, reinforced by an appeal to the continuity of the notation with that for hundreds, tens and units, or based on insecure rules for vulgar fractions. It is recommended that the rules be approached, or revised, through a practical need, which again may arise in Science, instead of being taught arbitrarily for use later.

In both decimal and vulgar fractions the crucial stage is the multiplication, and division, of a fraction by a fraction (6·246 to 6·248). This stage may well be postponed later in the course than other uses of the notations.

**5·34. The middle years.** It is in the choice of material for the middle years that the main purposes of the course will appear most clearly. Little can be added to what is said in other chapters about the claims of mathematical and practical topics.

*Number work* will be influenced largely by the choice of practical topics but all pupils who are able will make advance in the use of vulgar and decimal fractions as suggested in 5·331 above. The syllabus should provide for substantial progress in reckoning with numbers within the range of the pupils' experience and for extension of that range in type as well as in magnitude (6·15 to 6·152 and 6·21 to 6·216).

**5·341.** In the abler second or third year forms it may be necessary to arrange for transfers to other secondary schools (or to grammar school streams in comprehensive schools). If such transfers are expected the fact must be taken into account in deciding whether or not to teach certain numerical procedures to classes or groups of children before the transfers take place. Afterwards adjustments in the organisation of classes may be needed in the year depleted by transfers. It is possible to make these adjustments easier for those who have to be promoted by seeing to it that the mathematical ideas and notations involved are not strange to them even though some harder procedures may not have been learned in the lower class; for example decimal notation may have been used but the multiplication of decimals omitted.

Transfers may occasionally also influence the range of work undertaken in the beginnings of algebra and geometry (5·342, 5·344). The sections 6·3 and 6·5 deal with difficulties which may arise if the pace in these branches of Mathematics is in any way forced. Care must be taken to see that transfer requirements of a few able pupils do not involve the less able in work for which they have not the proper foundation.

**5·342.** Some study of *geometry* will be included in all courses. Many pupils of moderate ability come to enjoy this branch of Mathematics for its own sake, particularly geometry of a practical kind, and are therefore very willing to carry its study beyond the directly useful. Many suggestions, some of which are outside the limits of conventional courses of elementary geometry, will be found in 6·5 and chapter 7.

**5·343.** *Graphical work* is also much appreciated by young people. If begun in close relation to practical topics, it may lead to great advances in mathematical appreciation, again among pupils of moderate ability.

**5·344.** *Algebra* should be based on a wide number experience and is therefore not recommended early in the course, except that the use of letters to make general statements about numbers should be encouraged whenever appropriate. The procedures of algebra are unlikely to arise from practical needs at an early stage; the many who need them in technical work at a later stage will readily learn them if the foundations are sound (6·37).

**5·345.** If the practical work is appropriate, the notations of *trigonometry* may well make a first appearance in the middle years of a course. This is an attractive and not a difficult study in its early stages. The days are long past when trigonometry was regarded as a branch of advanced mathematics to be ventured upon only in the sixth form. The reasons which have made it popular much earlier in the grammar school course apply with equal force in the Modern School where simple work in numerical trigonometry will often be a suitable and convenient development of the later stages of the mathematical course.

**5·346.** It is specially important that the syllabus for the middle years should be viewed in relation to the course as a whole, for instance special consideration might be given to whether it provides adequate opportunity for the development of important general ideas such as ratio. Again care must be taken to utilise possible contacts with other subjects. Finally the middle years must provide a suitable foundation for the work likely to be undertaken in the last year of the course.

**5·35. The final year.** Although some Modern Schools are able to keep most of their pupils to the end of the fourth year or beyond, the loss of fourth year pupils term by term as they reach their fifteenth birthday often creates a major problem of organisation.

The course should reach some suitable conclusion at the end of the first term of the year so that leavers in that term have a sense of completeness. This applies also to the second term. It is however not contradictory to say that work in the final year may be arranged so as to give pupils the impression that there are new lines of enquiry open to them and new fields of usefulness for Mathematics, if they pursue their studies further. For example if logarithms are to be included in a course it will generally be wise to introduce them before the final year, so that pupils may gain some skill in their use and feel the advantage of lightened computation. On the other hand, if it has been unsuitable to introduce logarithms early, it may be of great interest and encouragement to the leaving pupil to discover the existence of such a powerful new tool even if he has not time at school to gain skill in its use.

**5·351.** Two separate factors may influence the choice of material for the older pupil, viz. his maturing social interests which may lead him to use his simple mathematical equipment in new ways, and his maturing mental powers which may lead him to mathematical studies for their own sake, for example deductive geometry and algebra. There are in addition in special circumstances vocational biases which may either focus interest on Mathematics or divert it away from mathematical study by the urgency of conflicting claims (3·5).

**5·352.** Some degree of individual study either within Mathematics or on a wider basis affecting all subjects may be possible in the final year. The ability to conduct some simple investigation involving the collection and use of mathematical data and perhaps the use of books should be encouraged for all pupils from the beginning of the course, but it should reach a climax in the final year and facilitate the arrangement of special studies for those who have special tastes and purposes.

**5·353.** Although the leaver's disposition towards Mathematics is an outcome of the whole course, it might well be an important function of the final term to combine a brief review of what has been done with a broadening of the field of interest. Pupils might constantly be challenged to investigate and interpret mathematical items culled from all everyday sources of press and radio and from commercial transactions and parental occupations.

**5·36. The least able pupils.** The number of pupils who are seriously behind their fellows because of poor intelligence, the dull, varies greatly from school to school. In some areas the provision made in Special Schools limits this tailing off for all other schools of a district, but the characteristic of the bottom 5 to 10 per cent of an age group is generally that the individual differences amongst the pupils is very great, not least in their capacity to deal with numbers. No syllabus can be made for even a small class; progress in Mathematics must be considered on an individual basis.

This does not mean that a class may not take part in some common project, each contributing according to his powers.

A few general principles may be stated:

(*a*) Many of these pupils have experienced discouragement over a long period. Remedial work can often be disguised as something new and exciting even though it descends to the easiest levels mathematically, (e.g. 4·3).

(*b*) Some few pupils seem to be unable to learn enough to free themselves from the use of apparatus for counting, adding etc. Great care is needed in dealing with such pupils and in diagnosing the extent of their dependence on reckoning aids. Pupils should be encouraged to build up their own table books and other ready reckoners for use as long as they are needed.

(*c*) It is wasteful of pupils' time and energy to dragoon them into doing calculations for which they do not know the use.

(*d*) The use of simple calculations of specific value to the ordinary citizen may sometimes be learned by rule of thumb, but generally it is possible to give some measure of understanding before practising a rule.

(*e*) Practice periods for mathematical skills, mental and written, should be quite short and the work should be individually graded.

(*f*) Limiting the syllabus to what is considered a bare necessity defeats its own ends. Variety, for example the simple study of graphs in interesting contexts and of geometry, has been found to favour development. If the link with the actual source of data for a graph is strong enough, even pupils of little ability may make progress in understanding the symbolism of graphs.

(*g*) Generally these duller pupils benefit greatly from integrated courses of study in which Mathematics is only one aspect.

(*h*) Pupils sometimes develop surprisingly about the age of 14 years, though the progress of others remains within the limits of the particular case and the concrete aid.

The least able pupils often find in the final year that the needs of everyday life spur them afresh to mathematical literacy.

### 5·4. Textbooks.

In some schools a textbook is made the basis of the scheme for a class or for a series of classes. This is an inversion of the ideal procedure that textbooks should be chosen to fit a scheme made to suit the particular circumstances of school and class. Reference to textbooks may however be helpful in planning a scheme.

Even those textbooks which consist only of practice examples have some claim to consecutive rather than incidental use. A well-designed book of this type does present a sequence of work, graded examples and suitably selected practice. A teacher who uses the book for occasional exercises, often the best use to make of it, should be aware that he may lose something of its value, since he becomes responsible for sequence and selection.

A few textbooks are written so that the pupil can read them and learn about Mathematics and its applications with little help from the teacher. These can more readily be used consecutively and as an adjunct to a course. They are very helpful to a teacher in his class-management since groups or individuals may work from them, from time to time, with only occasional direction. The teacher however should not fall into the temptation to use for his lessons what it is intended that the pupil should read for himself. With experience of using the book he should know at what stages pupils can manage alone. In a well-written book the reading matter is something more than an introduction to specimen examples. Even books of this kind are only a supplement to the teacher's own course.

### 5·5. Time Allowance.

The time allowance for Mathematics in a Modern School depends on a number of circumstances foremost amongst which is the amount and nature of practical work undertaken both within the mathematical syllabus and in the school as a whole. It is clear, for instance, that surveying, that is some form of outdoor measuring with appropriate instruments, is at some stages time-consuming, but it makes a contribution to education which cannot wholly be measured in terms of Mathematics. Again many applications of arithmetic studied with due regard for the importance of their practical background contribute much to general knowledge and perhaps very little to arithmetic. The claims of practical subjects themselves especially Handicraft, Housecraft and Science are often the determining factor in the number of periods allowed for Mathematics. That this is so is often shown by differences in time allowance for Mathematics for boys and for girls in mixed schools, where Handicraft takes up less time than Housecraft. The difficulty of arranging a scheme under these circumstances is great and the solution of excluding geometry from the girls' course is unsatisfactory. Again the nature of the practical work undertaken outside mathematical periods may influence what is specially interesting or specially needed as a balancing factor in a course.

The low allowance of three periods of 40 minutes per week, sometimes found for each year of the course, generally indicates that the school does not appreciate the contribution which Mathematics should make to the education of both boys and girls or that the school is so understaffed mathematically that advantage cannot be taken of any more time. On the other hand 5 hours or more per week generally indicates that there is some scarcity of practical work in the curriculum as a whole and that work within the mathematical syllabus may be expected to make good a good deal of this deficiency. The work in drawing connected with Handicraft undertaken by boys is a special factor; it has to be considered both as a valuable opportunity for spatial thinking and for its part in craftsmanship (7·3).

**5·51.** Differences in allowance of time in different years of the course are often convenient. Schools which find more time available for Mathematics in the first year because of the postponement of courses of Handicraft and Housecraft have to avoid the temptation of pushing on with mathematical techniques which are better suited to the more mature. Time may more wisely be spent in developing ways of working associated with all forms of practical mensuration, maps and diagrams and graphs of statistics, and in making secure *by varied use* the number accomplishments broached in the Primary School.

**5·52.** Differences of ability may also call for different distributions of time. Longer time may be allowed to a class because it is able enough to take advantage of additional opportunities and to develop the aesthetic aspects of Mathematics; or again a class which is less able may require longer periods for practical work and short periods for practice.

**5·53.** There is reason to plead for greater consideration for Mathematics than has been given to it in recent years in many schools on account both of its cultural values and also of its admitted utility. There are unhappily some schools, particularly amongst girls' schools, in which the teaching of anything but routine computation has diminished almost to vanishing point.

**5·54. Homework.** One consideration that may well be linked with the decision of time allowance for Mathematics is that of the place given to homework in the school. It is increasingly becoming the practice for some out of school work to be set. This does not invariably follow a traditional pattern; for example, ideas met within the classroom may lead to individual investigation outside, or statistical or other material may be collected for group discussion, see for example 3·324. In such ways the term 'homework' is being widened to cover what is done not only at home but at public libraries and elsewhere, including some at centres organised for this purpose.

**5·541.** Further thought is needed, both as to the scope and to the forms such additional work may most profitably take for pupils of different intellectual quality and at different ages, placed too in dissimilar environments. When an individual school makes its decision in the matter, it is desirable to ensure that any out-of-school activity contributes effectively to the development of the subject as a whole. Its ultimate value will depend not only on whether such work is regularly undertaken by all pupils, but also on how it is related to what is done in school. The successful integration of the two becomes a matter of class organisation (5·6).

### 5·6. Classroom Organisation.

Work in the mathematics classroom falls into three main categories, discussion and deliberation for the purpose of fostering of mathematical ways of thinking, practice of techniques with which the pupil is already familiar until they can be used without stopping to think, and work of a practical kind which is not necessarily in itself mathematical but which requires mathematical insight and the use of mathematical forms of expression.

The three types of work make contrasting demands on pupil and teacher alike. We now give some details of the kind of classroom organisation appropriate to each.

**5·61. Practical work.** We discuss practical work first because of its great variety. It is of special importance in dealing with the mathematically immature pupil who finds long excursions into mathematical thinking for its own sake difficult and unacceptable and who does not find verbal description of common experience a good enough foundation.

**5·611.** There are in the first place forms of expression used in Mathematics which are in themselves practical, viz. drawing, modelling and movement discussed in 6·12, 6·13 and 6·14. Although individual and group methods described below are appropriate to work in these fields, class lessons in which individuals do similar exercises and pool their results for discussion may be very profitable. For example the areas of circles of different radii might be found by different individuals by square counting and the results collected for class discussion and the drawing of a graphs Again the drawing of loci similar in type but with some numerical differences might reveal within a class interesting variety and particular case. (6·55 and Appendix 1).

The experimental production of loci suggested in 6·14 would be a matter either for individual experiment or for class demonstration.

**5·612.** Most practical work is for one reason or another best pursued as an individual or group activity. In the first place exercises calling for practical initiative are by their nature matters for individual effort or for group discussion with or without the intervention of the teacher. Again many exercises in measuring involve instruments which cannot be provided in sufficient quantity for use by members of a class simultaneously. The same kind of difficulty applies to the use of books and pamphlets for reference. The class teacher must therefore be prepared to organise work in a way which allows pupils individually or in groups to pursue different programmes at any one time but within a framework which provides that essentials are experienced by all in turn.

For some kinds of exercise a system of card assignments may be useful for individuals or for groups. It is no easy matter to draw up cards which provide enough advice to give pupils an adequate start but not so much that self direction is denied them. Too little advice on a card is to be preferred to too much. Textbooks may contain suggestions for practical work but, unlike the teacher's cards, cannot make use of the school environment. In 3·1 a combination is suggested of individual card assignments with grouped arrangements for giving experience of the workings of everyday money affairs. Card assignments are discussed in detail in 5·64.

Some practical work, for example surveying, requires absence from the classroom. It may at some stages be impossible to provide for all pupils to do outdoor exercises at the same time so that one part of the class must be provided with work which does not require close supervision. Fortunately there are many phases of mathematical work in which pupils best progress without active teaching; so that if a problem of supervision exists it is social rather than mathematical.

A different type of group organisation is envisaged in 3·3, in particular see 3·321 to 3·325

**5·62. Discussion and deliberation.** Few lessons and no series of lessons will pass without some class or group discussion. It is essential to good teaching to maintain a balance between stimulation by discussion and the pupil's individual efforts and experiment. It is unfortunately only too easy to allow the demands of a syllabus to leave so little time for discussion that class and group lessons become times for instruction, and individual efforts a mere working to rule without deliberation.

There are of course items of information, notations and conventions to which the teacher must introduce his pupils. It is important for the teacher to distinguish between what pupils can find for themselves and these arbitrary steps where his intervention is essential. Thus the choice of

90 degrees to the right angle, or 360 to the complete turn, and the writing of 10 × 10 as $10^2$ are arbitrary though their choice has a history, but it is possible 'to discover' that $10^2 \times 10^3 = 10^{2+3} = 10^5$ once the notation is understood.

**5·621.** A class or group accustomed to discuss what they do will go a long way towards answering one another's questions with little help from the teacher. His part will include bringing the class or group together when there is some item of practical or mathematical interest to discuss, some difficulty to be overcome or new field to be explored. He will also draw discussion together and add necessary or stimulating items to it.

A class which is unaccustomed to discussion of Mathematics generally finds it easier to talk about practical than mathematical topics. Thus a beginning might be made with group discussion of some problem in out-door measurement: how to align two lengths of measuring tape, how to measure or locate an inaccessible object and so on. Again discussion of whether the answer to some problem is a sensible one in the practical circumstances is at first more readily undertaken than discussion of the mathematical steps of the solution. Discussion of Mathematics which pupils have tried out for themselves follows as confidence grows.

Setting the stage for class discussion and enquiry can supersede a great deal of demonstration and description by teachers. Demonstration may appear to save time but even when it is understood it is easily forgotten.

**5·622.** There may sometimes be trains of thought which the teacher stimulates by a series of questions, but a form of discussion in which teacher's questions alternate with pupils' answers must be used with discretion. Such a series differs very little from instruction in the effect it has on initiative. Pupils of moderate ability very quickly lose the sense of sequence, so that in any long chain of questions the beginning is forgotten before the end is reached.

Question and answer is not in fact discussion, though it may stimulate discussion. When questioning is something more than incidental, it must be well distributed over a class and the effect of contact with the class as a whole achieved. The expected answer is not as a rule the most revealing: even a wrong answer may often be turned to good account.

**5·623.** Occasionally it is legitimate to guide pupils through a long series of steps to establish a general impression of the reasonableness of a result which is thereafter going to be used freely by the pupils. Such an occasion might be the explaining of a formula for the mensuration of one of the more difficult solids or the giving of a glimpse into the reasons for adding the logarithms of two numbers in finding their product.

Another useful type of questioning is that in which some idea is clarified by answering questions of increasing difficulty, often with the number content of the questions progressively harder. Such questions help pupils to generalise an idea.

**5·624.** Written problem work can often usefully be interspersed in discussion. To try out a problem or two gives pupils a feeling of participation and the teacher can give attention to those who need more help than others.

The very common practice of giving pupils a series of problems to which answers only are written may be seriously time-wasting, unless it can be arranged that the successful are well occupied while the unsuccessful discuss corrected results.

Problem work in arithmetic is discussed in some detail in the Associations *Mathematics in Primary Schools* (3·8).

**5·63. Practice.** In 1·5 we point out that emphasis on experiment and discussion is in conformity with the Association's *Mathematics in Primary Schools*, and that it forms the only sound basis for the practice of techniques. In the secondary stage any argument for rote learning based on the pupil's lack of time to master techniques important to everyday life before leaving school needs most careful examination for each individual. We consider that such a proceeding should seldom if ever be necessary (5·36 (*d*)).

**5·631.** Skill in using chosen significant mathematical techniques is one, but only one, of the aims of a mathematical course. Skill involves practice and practice can be obtained either in isolation or as part of the extended use of the technique. Generally extended use is to be preferred to long periods of isolated practice. Furthermore practice may be either oral or written.

**5·632.** Oral practice must be suited to the individual. It can seldom be that a whole class is a suitable unit for oral practice.

**5·633.** Written practice can without undue difficulty be chosen to suit the individual both in kind and in duration. Each pupil's need for accomplishing work of this kind makes it comparatively easy to break up classes into groups variously engaged. For example one group might be engaged on a surveying exercise while another completed drawings and went on to practice exercises.

Less practice is needed to master a technique by able than by less able pupils. Care must however be taken not to give less able pupils periods of practice too long for effective work; for them in particular practice through use is important.

**5·64. Card assignments.** Work cards may be introduced, as required, to control or extend individual written practice. Equally they may be adapted to promote topic studies, (3·12), or to permit a stocktaking, in a form more challenging than a routine test, of ideas and techniques separately mastered (5·8).

**5·641.** The various forms and functions of work cards are best determined for himself by each class teacher. He may, on particular occasions, devise series which will do one or more of the following:

(*a*) widen the context of a new idea,
(*b*) give assurance to a new technique,
(*c*) undertake a diagnosis of attainment,
(*d*) develop skill in extracting mathematical information from miscellaneous material,
(*e*) give zest to conventional topics by a more dynamic approach, see for example 3·1.

Work cards repay careful preparation, and grading in type and difficulty.

Among the main advantages of well-constructed examples is the encouragement given to independence of action, thought and reflection. By bringing together separate mathematical skills in a combined operation they help to make real a sense of the unity of Mathematics.

**5·642.** Though many work cards will suitably relate to the school's particular environment, others will usefully employ books, atlases, reference and other materials common to many schools.

In form and style cards may vary greatly; the ordinary postcard has proved in many years of use to be handy in size and of sufficient rigidity. To minimise the disruption of classroom organisation a typical card indicates briefly what materials, instruments and so on are required, but leaves the actual conduct of the 'job' to the pupil. Two examples follow, one of which is confined to the classroom.

*Work Card C12*

*You need:* Tape measure, geometrical instruments, compass.

*Your job:* Walk 25 yards in a direction 050° from the main school back door. Locate a wooden object. From it take bearings of the Church and of Cook's Tower. Sketch and measure the object, taking lengths to the nearest $\frac{1}{2}$ inch.

In the classroom
- (*a*) Plot the bearings.
- (*b*) Draw plan and elevation of the object to a suitable scale.

*Work Card G15*

*You need:* O.S. Map 100 (1 inch to 1 mile), geometrical instruments.

*Your job:* Flares sent up by a sinking ship are seen from A (Bidston Court) to be due West, and from B (Highton Station) to be 40° W of S.

From your map find the distance AB. Hence find how far the sinking ship is from each and from the New Brighton lifeboat station.

**5·65. The mathematics room and its equipment.** Schools differ in their use of 'subject' as opposed to 'form' rooms. Where form rooms are the rule for most subjects what is said of the mathematics room might apply to the form room of a teacher specially interested in Mathematics; other forms, particularly those of older scholars, might by arrangement occasionally use the room for their mathematics lessons. Where subject rooms are the rule throughout or in the upper part of a school, there should be at least one room which is always available for Mathematics whenever the subject is on the time-table, so that any group can by arrangement have the advantage of using it.

Whatever the organisation, storage space for equipment is needed.

**5·651.** The following points deserve attention in a mathematics room.

(*a*) Tables are to be preferred to desks for many activities connected with the subject, e.g. work with maps, charts, and drawing.
(*b*) Easily moved furniture is convenient for co-operative group work.
(*c*) It is often convenient to have a room with easy access to outdoor open space, e.g. for surveying.
(*d*) Blackboard space must be ample and a squared blackboard is required.
(*e*) Exhibition space should include shelves for models as well as space for pinning up diagrams and illustrations, including pictures and charts of matters of historical interest which may be both decorative and stimulating.
(*f*) Visual aids may on occasion include films, filmstrips and pictures or figures shown by episcope, if facilities are easily available.
(*g*) The room should have water laid on and a sink.
(*h*) A well-equipped room should include a workbench and tools; this is perhaps more important in girls' schools than in boys', where help can be obtained from the handicraft rooms.
(*i*) If mechanics is included in the scheme a suitably equipped practical room must include such items as a strong beam for carrying heavy weights.

Whether work is done in a subject room or in form rooms it is important for pupils to have places where work can be stored safely from one period to the next.

**5·652. Storage of equipment.** No modern school mathematics course can be pursued satisfactorily without storage space for the materials and apparatus involved in practical work. These are very varied but the following are some of the main categories:

| | |
|---|---|
| Drawing - - - - | - Boards, sets of instruments, suitable objects from which to make drawings (7·3). |
| Measuring apparatus - | - for various measures (6·8); for surveying, both standard and home-made equipment (7·18). |
| Models - - - - | - Objects and models of various shapes; home-made developable solids and their nets; globe, wire frame models etc. |

Charts and illustrations.
Logarithm and other tables.
Sets of books, reference books, pamphlets etc.
Special materials for remedial work and backward classes.
Apparatus made by teachers for pupils' use and for demonstration.
Pupils' work so long as it is of interest and use.
Handwork materials and tools for various kinds of practical work.
Mechanical devices and parts for constructing them, e.g. models with interesting moving parts.
Mechanics: if possible some real machines and machine parts as well as materials for a variety of mechanical experiments (7·4).

### 5·7. The Use of Books other than Textbooks.

Reference is made in various sections of the report to the use of books other than textbooks and occasionally the names of such books are mentioned. Some books on the history of Mathematics are listed at the end of chapter 8 and some useful charts and reference books for air navigation are referred to in 7·2. Other references to books and libraries, to pamphlets and to other sources of information occur in chapters 3 and 7, which deal with practical topics, and also in the section on practical work in this chapter (5·612).

**5·71.** The building up by a school of a collection of books about subjects of interest to young people in which some use is made of simple Mathematics is of great value in providing stimulus to mathematical studies. Teachers will find that even books above the pupils' mathematical level may make an appeal; provided that the notations used are not unfamiliar, pupils will take some of the working on trust and gain a useful impression of the wider scope of Mathematics.

The use of this literature of Mathematics, as it were, is a little explored field. The stimulus may come either from an interest within Mathematics to examine its applications, or, more probably at this stage, from an outside interest which is found to require mathematical expression.

**5·72.** The Association's *List of Books for School Libraries* also contains the names of reference books, more purely mathematical, many of which are at this stage more likely to be books for the teacher's use than the pupils'. Abler and older pupils, however, will often be greatly interested by being allowed access to suitable sections of books of a more advanced type than they would normally use.

### 5·8. Examinations and Testing.

Regular stocktaking of work accomplished is a normal feature of secondary school life. In arithmetic certain aspects of the work are very easily tested, indeed testing is a part of teaching method in dealing with the performance of simple routine procedures or with checks on items of information which should be remembered. When groups within a class have covered different ranges of work, tests should be on a group and not a class basis. Testing and recording is discussed at length in the Association's *Mathematics in Primary Schools* (chapter 5).

It is more difficult to devise suitable tests of a pupil's powers of dealing with an unfamiliar situation, but it is valuable occasionally to set pupils individual tasks in which they have to cope with practical problems, collecting their own measurements or other data required for a solution. The use of work cards in assessing attainment is discussed in 5·64.

It is also possible to test pupils' powers of comprehension of statements containing numerical or spatial information. The conventional 'mathematical problem' is posed in verbal form but it demands in addition to comprehension of its statement the power to work out a solution without concrete aids. It is in this respect a harder and less suitable test than the practical problem for many pupils.

Pupils' powers of making coherent statements about circumstances which involve numerical matter, spatial relationships etc. may also be tested, perhaps in the course of showing their understanding of some longer statement or of describing some situation presented to them practically or previously encountered. Such a statement is an exercise in itself and need not be accompanied by additional deductions.

The examinable content of work in geometry, in graphs etc. will vary greatly from course to course.

**5·81. External examinations.** The pupils with whom this report deals are those who have not shown particular aptitude for Mathematics at 11 years of age. Many of the ablest among them may at entry be suffering from pressure due to a selection test in which they were not successful. Any working towards external examinations early in the course is to be deplored.

At later ages a few pupils may mature mathematically in ways which make a late entry to existing school examinations an acceptable objective. The demands of these examinations in logical geometry and formal algebra are outside the scope of this report (5·342, 5·344, 6·3, 6·319, 6·37, 6·501 to 6·504).

It has been widely suggested that special examinations for Modern Schools might be devised, and in some areas these are already being tried. In Mathematics the danger is that such examinations would over-stress the easily examinable elements of courses and so tend to distort the courses themselves and to prevent them from serving the needs of individual schools. It is true that the Mathematics itself may differ little from school to school, but the process of learning is a protracted and varied one and the end points reached are mathematically few. A test in arithmetic only would be disastrous to a wider course such as we recommend. It is not easy even in examinations internal to the school to give the various elements of a course a balanced assessment. Moreover the wide range of ability covered by the term 'modern school pupil' makes any single examination inadequate for all pupils.

Some of the abler pupils may after they leave school take courses in Mathematics which are examined, for example technical and trade courses, nursing courses etc. It has already been pointed out in this chapter and in chapter 3 that in the final years of a 4-year course vocational incentives may well influence the choice of syllabus.

# PART II

## CHAPTER 6

## THE MATHEMATICS IN THE COURSE

### 6·0. The Scope of the Chapter.

The ordinary citizen requires only a few mathematical skills to carry on his daily affairs, but to keep in touch with what goes on in the world around him he must understand something of the language in which contemporary ideas are expressed. Much of this language is mathematical, and some understanding of it enables him to appreciate a good deal of what he reads and hears about modern scientific ideas.

**We consider it unwise that a pupil who is slow to acquire skill in computation should be restricted to efforts to do so. A more varied course will maintain interest and widen his background provided the computation involved is kept sufficiently simple.**

We have therefore brought together in a single long chapter all sorts of mathematical ideas and procedures which are within the scope of the ordinary pupil. We do not indicate which should be chosen in any particular circumstances, but discuss the development of the mathematical ideas themselves. The order of the subject matter and the language in which it is expressed are for the reader and not necessarily for the pupil, unless the context makes it clear that classroom presentation is intended. Teachers responsible for making syllabuses will find in this one chapter the parts of the subject which we consider likely to be suitable and relevant to the pupils' interests, while all will find comments on the mathematical implications of what they have to teach. We do not deal in detail in this chapter with the practical contexts; these are fully discussed elsewhere. Each main section of the chapter contains points applicable to various ages and stages. For notes on syllabuses see 5·3.

### 6·1. Mathematics as a Language.

For a secondary school pupil to make progress in mathematical thinking he must have opportunity to perceive relationships and truths and then to express them in ways of his own choosing, and not merely to reproduce them in the standard terms of more mature minds.

Mathematics is often thought of as a system of symbols written on paper, whereas there are several ways of expressing mathematical ideas, and we discuss these under five headings:

(*a*) words, spoken and written; (6·11, 6·18);
(*b*) drawings, both pictorial and diagrammatic; (6·12);
(*c*) modelling in various materials, including cutting and folding paper; (6·13);
(*d*) movement; (6·14);
(*e*) chosen symbols such as numerals, letters and signs whose forms are traditional; (6·15 to 6·18).

**6·101.** The entrant's experience of these modes of expression may be very limited. His drawing, modelling and movement are unlikely to have been associated with mathematical ideas, though they have given him some elementary appreciation of space relationships.

In what follows we shall assume that the children have only slight knowledge of mathematical language, particularly where symbols are concerned. Primary Schools are however taking more account of mathematical ideas and the way they are expressed, so that some pupils may be better informed. Entrants probably understand the use of numerals to denote numbers which are small enough to be easily comprehended, but the signs + − × ÷ are likely to be to them merely instructions to carry out certain arithmetical operations, so that $427+936$ is an *exercise* to be worked out rather than just a *number*, and $a+b$ may seem at first to be meaningless because it cannot be worked out. Full understanding of these signs would be shewn by an ability to use them in appropriate contexts.

### 6·11. Words.

A young child first expresses his mathematical ideas and operations in words, but different verbal statements, expressing different real life situations, may lead to the same statement about the numbers involved. Thus sharing 20 sweets among 5 children and finding how many 5 shilling books can be bought for 20 shillings both lead to 20 *divided by* 5. This is often a source of difficulty especially when pupils become accustomed to treating, for example, the word *share* as though it could always be used in place of the word *divide*, as they often do when referring to a division as a 'sharing sum'.

**6·111.** Many words and phrases have been taken from ordinary language and given precise mathematical meaning, such as *obtuse*, *coordinate* and *index*; while many of mathematical origin have been adopted for non-technical usage, usually with the same fundamental though less specific meaning, for example *square*, *momentum* and *diametrically opposite*.

Thus the language of Mathematics is closely linked with the living and growing language of daily speech, and should be intelligible to the ordinary man, but the use of the same words with ordinary and with precise meanings is not likely to make for understanding.

**6·112.** A pupil's vocabulary is acquired largely at home and may reflect his environment by including words peculiar to the neighbourhood or to the occupation of a parent. Thus a farming community may use measures not in urban use and the technicalities of electrical trades may become familiar to younger members of a family. A teacher may well find it necessary to increase his own knowledge of words and technical terms commonly used in the district of the school, and also to help his pupils to a clearer understanding of them.

We recommend

(*a*) That note be made of the pupils' use of words having quantitative, spatial or technical significance, and that the refining of meaning be achieved by making later opportunities for their exact use.

(*b*) That words not so commonly used in everyday speech, such as *diagonal*, *triangular*, or *perpendicular*, should be introduced in describing some situation in which the pupils are actually engaged. The pupils would then be expected to use the words in suitable contexts rather than to define them formally. In some cases, however, such as *acute angle*, a precise definition will aid the understanding and proper use of a term.

Accurate verbal description should be considered an essential part of mathematical training, as should also the correct following of instructions given in quantitative or spatial language. On the other hand unnecessary technical terminology should be avoided. Words which it is desirable for a teacher to know for his own reading are not necessarily suitable for his pupils. Thus some words in this document are for teachers rather than their pupils.

(*c*) That attention should be drawn, in suitable cases, to the similar or contrasting use of words or phrases in technical and in everyday language; e.g., *acute*, *perpendicular* (with its architectural significance), *parallel* in the sense of *analogous*, and *multiply* (with its biological meaning).

(*d*) That the association of words with more concise forms of mathematical expression be developed progressively throughout the course, so that a pupil will learn to speak of a *right angle* rather than a *square corner*, of a *quotient* instead of *the answer to a division*, will say that 13 is a *factor* of 1001 instead of saying that 13 *divides exactly into* 1001, and will write $AB=AC$ instead of the equivalent statement in words. The reverse process of interpreting mathematical symbolism is referred to in 6·114.

(*e*) That attention should be drawn to the derivational connection between mathematical words and others, either technical or non-technical, the pupils being invited to add further illustrations; e.g., *poly-gon*, *poly*-technic, hexa-*gon*, *dia-gon*-al, *dia*-meter, *tri-gon*-ometry, *tri*-angle, *tri*-partite, *cent*-ury, per *cent*.

**6·113.** While pupils should become more and more able to express mathematical ideas in conventional manner and terminology, it is not necessary, or even desirable, to dispense entirely with less formal modes of expression. Thus the pupils should know the words *numerator* and *denominator*, but should still be allowed to speak of the top or bottom of a fraction. The more formal language is customarily associated with the written rather than the spoken word.

Nevertheless incorrect use of words must be promptly dealt with. Examples of common errors are: 'I times the length by the breadth'; 'straight' for upright or vertical; and 'proportion' for 'ratio'. To allow an angle to be described as a corner is to lay up trouble for the future.

It should be explained why such expressions are wrong and the correct forms made familiar to the ear by repeated, though not laboured, use of them.

**6·114.** The pupil should frequently use ordinary language to interpret mathematical statements, as in saying in words what is expressed by a formula or describing how some calculation has been performed. Sometimes the mathematical language will be geometrical, and the verbal description a statement of the geometrical relationships involved in some diagram. Both comprehension and expression are involved in this interchange between verbal and symbolic expression.

In the paragraphs which follow we shall, when appropriate, take each discussion back to its verbal beginnings.

### 6·12. Drawing.

Drawing, together with modelling (6·13), is essentially the language of geometry, and as such it is fully dealt with in other sections where geometrical expression goes hand in hand with geometrical discovery. The power to make and to read a sketch which reveals geometrical relationships is sometimes overlooked as a teaching aim in pursuing the niceties of draughtsmanship and scale drawing.

It must not be assumed that a pupil has a proper conception of such geometrical ideas as that of an angle or the equality of lengths until he has shewn that he can both make and understand statements which involve them.

But the use of drawings goes far beyond the bounds of geometrical thought. The appeal of the visual to most adults, and to children who have passed the early stages of spatial appreciation, makes a language based on drawings of particular significance; indeed it would hardly be too much to say that those mathematical ideas which can be given a diagrammatic form are accessible to the ordinary man, but that those which cannot may well remain a mystery to all but the specialist.

The diagram is mainly useful in clarifying a situation, though not infrequently it may be the means of suggesting how to solve a problem.

Diagrammatic forms of symbolism develop alongside the purely symbolic forms. Like the spoken and written word they can be treated as a stage towards the learning of other symbols, and as such they will be noted in discussing those symbols. For instance, lines or columns of different lengths, representing numbers, are helpful in the discussion of the natural numbers, of fractions, of ratios, and so on, and also in the representation of quantities of all kinds, including lengths themselves as in drawing to scale.

Graphs, which form a particularly important class of diagram, are dealt with in detail in 6·4.

### 6·13. Modelling.

The term modelling is intended to cover any demonstration with material objects, other than drawing on a flat surface, for example paper-folding. Modelling is most commonly used in clarifying spatial concepts: symmetry by paper-folding; equality of areas by dissection; the properties of solids by models, frameworks and developments; congruence by paper cutting and so on. Such examples are discussed in 6·5.

Diagrams may also have adjuncts, such as adjustable strings and three-dimensional extensions, which bring them into the realm of modelling rather than drawing. Dissections for illustrating the properties of fractions are a simple example. Often a model is made and used for some purpose other than mathematical demonstration. The teacher of Mathematics can take advantage of the occasion, either to point out its properties or to help overcome difficulties of construction. For instance in making models of buildings right angles have to be dealt with, and also perhaps awkward angles between planes. Dressmaking may raise questions of symmetry and of similar figures; in carpentry interest may be aroused in the nature of a screw thread and the contrast between left-handed and right-handed threads.

### 6·14. Movement.

Movement as a form of expression also has its main uses in dealing with spatial concepts and the line between it and modelling is rather hard to draw; for example folding may be considered as a movement, and curves traced by mechanical devices involving rods and strings as the product of models.

Although the time element necessarily comes into demonstrations based on movement, it is, except in the study of movement itself as a part of mechanics, usually disregarded in favour of the purely geometrical aspect. Thus the angle concept is most readily appreciated in relation to a turning movement from a fixed line (6·511). In illustrating a right angle and two right angles by 'right turn' and 'right-about turn' the time taken to make the turns is not considered. Again the idea of a locus (6·55) is appreciated best as an actual track made in moving, but the time taken to trace the locus is no part of the concept.

Some mathematical films and film-strips are available which exploit movement as a means of carrying conviction to the pupil. Angles may be swept out to draw attention to their placing, for example an angle of latitude in a hollow sphere may be first drawn and then rotated about the polar axis so that a point on the sphere traces out the small circle of latitude. A rolling wheel may be shown with a marked point on it, which in a second showing appears to trace its path. Again, figures the same in type but changing progressively in their quantitative relationships may be shown in series, for instance circles with their common tangents as their centres approach one another.

Graphs of the distance-time type may sometimes be built up very effectively by experiment. For instance a pendulum carrying a marker will trace out a sine curve on a paper pulled steadily across the swing. In 6·42 (*a*) the continuous temperature recorder is an example of a slow moving trace.

### 6·15. Number Symbols and Extended Ideas of Number.

Our ten numerals 0, 1, 2, . . . 9 are symbols, whose significance a child can grasp from his early experience. For higher numbers we do not go on inventing new symbols; we have a system whereby these higher numbers can be denoted by use of our original ten symbols. Such a system is called a notation. It is true that young children often learn to recognise the smaller numbers such as 12 or 15 as if they were independent symbols and without appreciating the fact that they are notational expressions for $10+2$ and $10+5$ respectively. But a proper appreciation of larger numbers such as 758, and ability to compute with them, must depend on understanding the additive nature of the notation. Such understanding is usually assisted by expressing the number in words, as seven hundred(s) and fifty-eight, though the unfortunate reversal in the 'teens', where for example the 'four' in fourteen precedes the 'teen', may confuse rather than help.

Diagrams or apparatus shewing a number built up of units, tens and so on may help appreciation of the notation. The abacus or counting frame is an example. Pupils need help of this kind to understand the importance of writing 0 to correspond to an empty column.

In work on factors of numbers a class will meet such examples as $32=2\times2\times2\times2\times2$, and will readily realise the advantage of an index notation in which the result is written $2^5$. Squares and cubes, which can be so useful in the learning of area and volume, may soon be used quite competently, as may also powers of 10, which are an essential part of our decimal system of notation; but on the whole the index notation requires long use as a form of expression before attempt is made to build upon it.

The Roman system of numeration must of course be dealt with, not only for its historical interest but because it is far from extinct. Other systems of historical interest might be described and discussed in the light of our present decimal system. The term decimal system refers to our notation in which we count in powers of ten, and must not be thought to refer simply to our system of 'decimals' in the sense of decimal fractions.

**6·151. Fractions.** Methods of indicating fractions, both less than 1 and greater than 1, form a very important part of our mathematical vocabulary, and it is as such that they are considered here; methods of teaching the use and manipulation of fractions are dealt with elsewhere.

When fractions are first introduced it is recommended that special words for describing them such as *vulgar* and *improper* should not be used. It is generally better to use no prefix at all in place of vulgar and to call fractions greater than 1 by some such name as top-heavy fractions. Thus at first $\frac{2}{5}$ is simply a fraction and $\frac{7}{5}$ a top-heavy fraction if any distinction is needed.

When decimal fractions are introduced the word fraction without the prefix decimal can be used for any fraction other than a *decimal fraction*, from which the word fraction will soon be dropped. If greater emphasis on the distinction of type is required perhaps the word ordinary might replace vulgar. In this report the time-honoured words vulgar and improper are retained in circumstances in which pupils might use ordinary and top-heavy.

A child's first experience of a fraction is as a means of indicating a part of a whole; the denominator and numerator show how the whole is divided up and how many parts are taken. The impossibility of having five quarters of a particular cake tends to give the idea that a fraction cannot be greater than unity, but such examples as five quarters of a pound of tea or the common expression 'three halfpence' should lead on fairly easily to appreciation of improper fractions. Mixed numbers are readily understood, but stress should be laid on the equivalence of these numbers and their expressions as improper fractions, and on such equalities as 1 lb. 3 oz. $= \frac{19}{16}$ lb. Diagrams are of great assistance in such discussions.

Just as a piece may be expressed as a fraction of the whole, so one quantity may be expressed as a fraction of another quantity of the same kind, although it does not form part of it. Such fractions may well be greater than unity, and are then often written as improper fractions to give a more vivid mental picture of the comparative sizes of the two quantities; such fractions, whether greater or less than 1, can be regarded as *ratios*. A ratio can also be written in the notation 3:2 (6·183).

There are other notations for fractions, notably percentages and decimal fractions. 'Shillings in the pound' is used in special contexts, but is generally considered as a *rate* (6·182).

*Percentages* are very frequently used where the values of a number of fractions are to be compared, perhaps with some implied standard as in marks, attendance and so on; and also to express a profit or loss, or interest paid as a fraction of the sum of money involved. They are almost ideal for indicating how a whole is divided into parts; e.g. how the local rates are used for various purposes, or how a mixture of solids or liquids is made up of its constituents. A circle divided into sectors is an excellent type of diagram for illustrating such percentages (6·512); but percentages above 100% are much used, and must be given due consideration.

Historically percentages preceded the notation for decimal fractions of which they are a special form. The linear scale showing units and tenths forms an indispensable diagrammatic approach to the appreciation of the first place of decimals with estimation of the second place as tenths of a tenth. This is on the whole more important than the idea that two places of decimals give the number of hundredths. This latter is often shown diagrammatically by taking a square inch divided into 100 square tenths or hundredths of a square inch. Thus 0·37 sq. in. is given by 37 of the small squares or 37 hundredths. But this interpretation alone does not sufficiently emphasize the importance of the leading figure.

In Britain decimal fractions suffer almost more than any other part of arithmetic from being taught as a notation detached from experience and without adequate foundation of a 'descriptive' kind. This is due to the nature of our weights and measures, few of which are related decimally. The use of tenths of an inch in drawing and measurement and in work on squared paper gives opportunity for decimal expression without heavy computation. It is important that as many pupils as possible should learn to read decimal fractions and to write them with understanding in appropriate circumstances. On the other hand much computation with decimals is for many pupils unimportant (6·247).

Clear notions about the decreasing significance of the digits in a decimal notation is an important preliminary for any who consider *irrational numbers* such as $\sqrt{2}$ or $\pi$.

The development of the idea of a system of fractional 'numbers' is greatly assisted by the diagrammatic representation of fractions on a line. Other types of diagram are discussed in 6·241.

**6·152. Other extensions of the idea of number.** There are topics which may be included in a syllabus and which give opportunity for the introduction of the symbolism of *directed number* e.g. temperatures of 15° above and below zero on the temperature scale expressed as $+15°$ and $-15°$. Again the terms 'Zero Hour' and 'Zero Hour plus (or minus) one' etc. have entered common speech from the timing of wartime operations. The meaning is generally clear from the context, but this example is but one of several instances of the use of directed numbers falling within ordinary experience. Speaking diagrammatically the characteristic of such numbers is that they have a right and left property in relation to a chosen point on a line drawn from left to right, or the equivalent in any direction. *Zero* is the number name for the chosen point or *origin*. If times are denoted by the numbers, the right and left property corresponds to after and before zero hour. It is usual to denote the right and left property by *plus* and *minus*, a use of signs totally different from their use as indicating sum and difference (6·29). The plus direction from the origin is conventionally noted by an arrowhead.

Discussion of quantities which can clearly have positive and negative values, such as temperature, time, height above sea-level, rate of flow of water into or out of a tank, and even speed and force, with the use of suitable terminology and linear diagrams, forms a comparatively easy activity which leads on to an extended study of graphs and opens up a number of important ideas especially to those whose mathematical interest may be aroused later.

Another field of enquiry into which a first step may be taken diagrammatically without much difficulty is that of *vector* quantities, such as velocity, which require both magnitude and direction for full description. The movements of aeroplanes carry the imagination out into displacements in three dimensional space. The representation of these displacements by a model to scale is like the modelling of an object except that we have the model of a journey rather than that of something we can see all at one time. The next step, that of recognising that there are other quantities which share with displacement and velocity the property of having both size and direction, depends for its success on the vividness and familiarity of the practical situation examined. Force and displacement are probably the easiest vectors to treat experimentally in the classroom.

**6·153. Generalised number.** The use of a letter of the alphabet to represent an unknown number, or 'any number', is not in itself a difficult idea for a pupil who knows what he means by a number even though his idea may be a limited one, e.g. may not include directed number.

The difficulty in generalised arithmetic often arises in trying to make statements, in mathematical symbols, expressing relationships between the numbers and letters involved. These mathematical statements are far superior for the usual purposes of Mathematics to the corresponding verbal statements; but they generally involve the use of the signs +, −, ×, ÷, to indicate the *results* of adding, subtracting, and so on, and not as *instructions to perform these operations*. This use of the signs is at first unfamiliar.

Moreover the ability to make such a general statement demands a complete mastery of the situation. Take for example the simple case of bringing shillings and pence to pence. A pupil may say 3s. 5d. is 41 pence almost without stopping to think; but if he has to bring *a* shillings and *b* pence to pence, he must bring the process used out into the open again and say: first multiply the number of shillings by 12 giving $(a \times 12)$ pence, then add in the *b* pence giving $[(a \times 12) + b]$ pence.

It is recommended that the form $a \times 12$ be allowed at first and replaced by $12a$ (as more convenient) as soon as the implied multiplication in *twelve a*s can be relied upon, as it is in *twelve fours*.

Furthermore any demand for formulation should be introduced gradually over a number of years as pupils gain command of one situation after another. Many will hardly use generalised statements at all and few advance safely much further than the direct use of a formula made by the pupil. A detailed discussion as to how far algebra can be introduced will be found in 6·3.

### 6·16. Statements about Numbers.

It may be appropriate to summarise here what has been and will be said about the signs +, −, ×, ÷, as parts of mathematical language. It has been pointed out that much has to be done in the Modern School to fill out their fundamental meaning applied to the natural numbers. These signs have no meaning apart from numbers and in 6·244 there is discussion of how their meaning is extended to suit the extension of the idea of number to include fractions. Further modifications are required when directed numbers are considered (6·29). Statements using these signs, and others such as =, are made about numbers, and the signs should not really be used when the statement is made about objects; e.g. $36 \div 6 = 6$ is correct, but, pedantically, the statements 36 sweets $\div 6 = 6$ sweets, 36 in. $\div 6 = 6$ in., are incorrect, though an accepted, extension of the symbolism. The full verbal statement of the situations all of which involve $36 \div 6 = 6$ depends on the nature of the situation which may be of the kind known as ' measuring ', 36 in. $\div$ 6 in. $= 6$, the number of the 6 in. lengths in 36 in. is 6, or the kind known as ' sharing ', 36 sweets (or inches) divided into 6 parts gives 6 sweets (or inches) in each part.

There is generally more than one suitable form of a number statement e.g. $5 + 3$ may be read as an instruction ' add 3 to 5 ' or as a whole ' the sum of 5 and 3 '.

It is important, but not easy, to encourage and refine verbal statements about numbers without unduly limiting the spontaneity of their form. The noting of crudities for correction later by example is often better than an immediate reduction of the statement to correct adult form.

In symbolic statements, however, the various signs and symbols must be used with rigorous correctness.

The sign = is one of the most frequently misused. It expresses the quantitative equality of the expressions between which it is set and corresponds to such a phrase ' is equal to ' or ' has the same value as ' in the verbal statement. A very common error is to use it between consecutive statements of an argument, in the sense of ' therefore '. One statement cannot *equal* other statement. See 6·35 also.

The sign $\equiv$ is properly used to indicate that two expressions are *identically* equal. Thus we have, for example,

$$2x+5=11 \qquad \text{(an equation)}$$
$$2(a+b)\equiv 2a+2b \quad \text{(an identity)}$$

The statement '$\triangle ABC \equiv \triangle PQR$' means that the two triangles are congruent i.e. equal in all respects.

When several numbers are separated by the signs +, −, × and ÷ brackets should be inserted to show in what order the operations are to be performed. Numerical expressions of this kind are unlikely to occur except in situations in which there is no doubt about what is intended, for example in finding the area of two walls of a room when it is convenient to add length and breadth before multiplying by the height, or in evaluating a formula in which numerical values are given to an expression such as ($ab$ + other terms). Artificial examples should be avoided unless it becomes necessary to emphasise that the order of operations is important. Thus

$$(8-3)\times(2+10)\div 2=30$$
$$8-(3\times 2)+(10\div 2)=7$$
$$[(8-3)\times 2]+(10\div 2)=15$$

If after leaving school pupils work in some sphere in which a convention about the order of operations has been adopted, they will learn it the more readily for having been trained to recognise that this order is of significance. We deprecate the testing of school boys and girls to see if they have learned such a convention.

The use of brackets is considered further in 6·314 and 6·38.

Mathematical symbolism is superior to verbal expression in that it leaves no ambiguity. To write that 'the radius of a circle is the square root of its area divided by $\pi$' is ambiguous; the statement could be clarified by rewording, and the spoken statement made clear by pausing, but the mathematical statement $r=\sqrt{\frac{A}{\pi}}$ is unambiguous.

These two examples of mathematical symbolism, the = sign and brackets, are in universal use, and must be known. Many others, such as ∴ for *therefore*, can be introduced at the teacher's discretion. The rule is always scrupulous correctness of usage, and adequate time to gain familiarity in simple contexts. Misuse of signs might well be penalised, in some cases, by insistence on the use of the verbal equivalents. Some further details of symbolism are discussed in 6·38.

### 6·17. Other Notations.

There are other important mathematical notations, which may be introduced in some Modern Schools according to the nature of the course attempted. For example it is suggested later that the use of logarithms is within the scope of pupils of quite modest ability and that it may conveniently widen the pupils' range in computation. This word *logarithm* will probably be employed with partial understanding as applying to numbers found in a table and used for performing calculations.

The term *tangent of an angle* however, may, if the course includes practical work of a suitable kind, be used with real understanding of what it denotes in limited and particular circumstances. The symbol *tan A* will be used as a convenient shorthand in making statements about a right-angled triangle. The strangeness may also be removed from statements which involve other trigonometrical ratios.

### 6·18. Some Important Mathematical Terms.

Many important mathematical ideas are summarised by words whose meanings grow only slowly as both practical and mathematical experience is enriched. Three such are chosen as examples: *average, rate, ratio*. The first two are in common use colloquially, the last only occasionally, but it belongs to a most important mathematical notion, that of comparison. For the ordinary citizen it is at least as important to have a clear idea of the relationships expressed by these words as it is to know how to find an average, a rate and a ratio; but the ideas expressed can only be given precision if there is at least some simple experience in computing as well as of using the words in a correct context. ' Using ' and ' finding ' may include geometrical examples and diagrammatic methods, which are particularly effective in making an idea clear. The actual calculations and graphical work are dealt with in other sections.

For example finding an average is discussed in 2·131; a rate of change is, qualitatively at least, associated with the interpretation of a graph at an early stage (6·42*b*); the comparison of two heights may bring out vividly that comparison by ratio and comparison by difference serve different purposes.

Opportunities for using these terms will occur again and again in any course which has real significance for the pupils. Some situations in which the terms might be used are listed below:

**6·181. Average.** Contexts in which this term is used involve statistics of some quantity which changes from individual to individual instance, such as heights of individual people, plants, etc.; number of eggs laid week by week; scores from innings to innings, trial to trial etc.

The average of a continuously varying quantity, such as the average speed of a car for a certain journey, is a much more difficult idea than the average of a finite set of values, and needs separate and careful treatment in so far as it is dealt with at all. On the other hand, the phrase ' average temperature for July ' probably does *not* indicate the mean value, but the mean of a *finite* number of temperature readings.

**6·182. Rate.** A rate gives the amount of one quantity which corresponds to an amount of another quantity as in 2s. 6d. in the £1. This is a *uniform rate*, but in life exactly uniform rates are the exception. Nevertheless the idea of an *average rate* is very simple and plenty of examples will come readily to mind, for example

200 words per minute
5 houses to the acre.

It is usual to cite one unit for the second quantity; thus ' 32 stitches for an eight-inch band ' is a rate, but it would usually be simplified to ' 4 stitches to the inch '. A birth-rate on the other hand, is usually given in the form '21·2 per 1,000', where the 1,000 can be thought of either as a convenient round number where 1 would be unsuitable, or as itself a unit.

A non-uniform rate is a very much more difficult idea, though in this age of travel the phrase '20 m.p.h. at the instant' probably has a fairly definite significance to most children. Graphs showing the relation between pairs of varying quantities such as distance and time, or barometric pressure and time, help to clarify the idea of speed at an instant, or of rate of change of barometric pressure.

**6·183. Ratio.** Variously worded expressions occur when the notion of the ratio of two numbers is concerned. For example:

(*a*) 'twice' and 'times' as in 24 is twice 12, or three times 8, the ratios being 2 to 1 and 3 to 1.
The ruler is twice as long as the pencil, or the ratio of their lengths is 2 to 1, which is also written 2:1.

(*b*) fractional statements, as in 12 is half 24 and 16 is two thirds of 24 giving ratios of 1 to 2 and 2 to 3. Again these can be written 1:2 and 2:3. 10 per cent or a ratio of 10 to 100 or 1 to 10, since 10:100 is the same ratio as 1:10.

(*c*) equal ratios giving four numbers or quantities *in proportion* as in 16 is to 24 as 2 is to 3; or the sides of the two figures are *proportional* meaning that ratios of corresponding sides are equal. This last example occurs in all 'similar' figures and therefore in drawing to scale, enlargement, etc.

(*d*) division of a line in the ratio *m* to *n*, e.g. 4 to 7; diagonal scales; vernier.

(*e*) in mechanics, velocity ratio, gear ratio, compression ratio of engines.

(*f*) specially named ratios, e.g. the sine of an angle.

(*g*) important ratios e.g. ratio of circumference to diameter of a circle.

(*h*) the notation 4:3 for a ratio of 4 to 3 is often extended to the comparison of three quantities, 4:3:1, so that if these 3 quantities are constituents of a mixture they are severally $\frac{4}{8}$, $\frac{3}{8}$ and $\frac{1}{8}$ of the whole.

A ratio expresses, not the difference between two quantities *of the same kind*, but how many times one is contained in the other, or the measure of one when the other is taken as unit. Thus to say that two distances are in the ratio of 5:2 can be thought to mean that, if the second is taken as a unit of length, the first is $\frac{5}{2}$ units. But for most people the implication is even simpler: that, if a suitable unit is taken, the two lengths actually *are* 5 units and 2 units, just as two lines might be 5 inches and 2 inches. The ratio of two quantities gives no indication of their magnitudes, since the common unit is not stated.

Ample experience is needed of this means of comparing the sizes of two quantities before it can become part of a child's mathematical vocabulary. The idea of ratio grows slowly, and patient treatment is necessary.

## 6·2. Numbers and the Basic Processes.

*6·21. Number knowledge.*
*6·22. Calculation with small integral numbers; tables.*

*6·23. The four rules for integers greater than ten.*
*6·24. Fractions and calculation with fractional numbers.*
*6·25. Other methods of computation.*
*6·26. Checks.*
*6·27. Approximate calculation.*
*6·28. Logarithms and the slide rule.*
*6·29. Computation with directed numbers.*

This important section is concerned with the extension of the pupil's knowledge of numbers and his powers of calculating both mentally and on paper.

### 6·21. Number Knowledge.

In the Primary School the pupil will normally have learnt to use the smaller numbers, say up to 100, with confidence and accuracy, and to know their properties in some detail. The consolidation of such work is therefore dealt with in the remedial section (4·4 and 4·5).

Here we consider some of the interesting properties of numbers, both within and beyond this range, which will appeal to all pupils.

**6·211.** Certain kinds of numbers have particular characteristics which are worth studying, for example, odds, evens, primes, squares, cubes.

(*a*) 11, 59, 137 cannot be divided without remainder by 2 or any larger number, except 11, 59 and 137 respectively; they are odd and prime.

(*b*) 36 is even and a square
121 is odd and a square

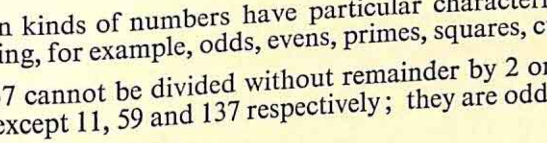
(*c*) 27 is odd and a cube
64 is even and a cube

The pupils can discover others of similar types.

Numbers can form a series which the class can continue, recognising the way it is built up, e.g.

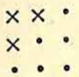
1, 5, 9, 13, ......
2, 6, 18, 54, ......

Certain numbers are connected with patterns, e.g. the triangular numbers.



*Note :* The series 1, 3, 6, 10... is a specially interesting one and occurs again in building up the square numbers:

$1+3=4$
$3+6=9$
$6+10=16$
........



or, writing in series form, as below.

$$\begin{array}{cccccc} 1, & 3, & 6, & 10, & 15 & \ldots\ldots \\ & 1, & 3, & 6, & 10 & \ldots\ldots \\ \hline 1, & 4, & 9, & 16, & 25 & \ldots\ldots \end{array}$$

Discovering such properties involves only counting but has the value that it helps pupils to become enjoyably aware of number relations.

**6·212.** Some numbers are remembered for their uses and association, particularly in relation to measures of such things as time and length. For example,

22, 440, 1760 yards form a group.
240 pence = $(12 \times 20)$ pence.
12 in. = 1 ft., 144 sq. in. = 1 sq. ft., 1728 c. in. = 1 cu. ft.
360° = 4 rt. angles = $(4 \times 90)$ degrees.
60 min. = 1 hour = $(5 \times 12)$ minutes.

**6·213.** The uses of number other than counting and computing are interesting. The numbering of pages in a book or houses in a street gives each a label as well as enabling a count to be made. On the other hand a telephone number or a car registration number may contain a figure which defines a district or a date. It is not used therefore as a true number; in such a case 673 could properly be read as six seven three and not as six hundred and seventy three. The study of such uses can lead to a better understanding of number as a sequence arranged in a scale of tens (4·31).

**6·214.** The decimal notation itself, units, tens, hundreds, etc., provides the number pattern of greatest usefulness and leads to the discovery of other patterns worth remembering.

For example, adding nines continuously, beginning with any number, gives a series with the units figure decreasing by one. If 8 is the starting point the series is

8, 17, 26, 35, 44, ......

The test of divisibility by 9 also depends on the tens basis of our numbers: 72 *must* be divisible by 9 because

7 tens equal 7 nines and 7 ones
$\therefore$ $72 = 7$ nines and $(7+2)$ ones.
$7+2=9$
$\therefore$ 72 can be written in nines.

The study of a few more examples will lead to the rule ‘ add the digits; if their sum is divisible by 9, so is the number ’.
*Example :* 76923.

$7+6=13$. Cast out a 9. This leaves 4.
Ignore the next figure, a nine.
To 4 add 2 and 3.
This gives 9.
*Or simply* $7+6+9+2+3=27$, $\therefore$ 76923 is divisible by 9.

This statement should, of course, be checked by the pupils. But 76913 is *not* divisible by 9, as can readily be checked by applying the rule and also performing the division. A grasp of the number properties used in such devices greatly strengthens confidence in calculating.

**6·215.** The ideas underlying a decimal notation have been discussed in 6·15. They are worked out in detail in the remedial section (4·3). We stress here the need for all pupils to have practice in naming correctly any of the component parts of a number written in the decimal scale. For example, in 3529, he should recognise 3000, 20, 500, 9, and be able to write the number in such a form as 35 hundreds and 29 units, or 352 tens and 9 units. At a later stage he will be able to re-write it as 3·529 thousands (6·247). The ability to think of a number as the sum of its components, 29 = 20 + 9, is necessary for the use of ready-reckoner tables as well as for understanding the composition of numbers with more than two digits for the purpose of long multiplication.

**6·216. Large numbers.** Large numbers are awkward to write and to read when set out in full; nor is their significance or comparison easily seen. Expressed in terms of a large unit, say millions or thousands, they become more manageable. Writing 2,500,000 as $2\frac{1}{2}$ millions, and similar exercises will be found useful at this stage, but it is better to defer writing such a number as 17,469 in thousands until the decimal form 17·469 can be readily used. Graphical ways of representing larger numbers are valuable before decimal methods are undertaken. The prefixes *Kilo* and *Mega* are becoming increasingly familiar.

**6·217.** By giving a group of numbers diagrammatic form as lines, columns or symbolic pictures (isotypes), important comparisons can be made. For example, suppose statistics involve 350 men, 200 women and 400 children. These could be represented by lines on a scale such as 100 persons to each inch. Or they could be represented by picture graphs in which each picture, a symbolic man, woman or child, represents 100 persons, as shown in Figure 1.

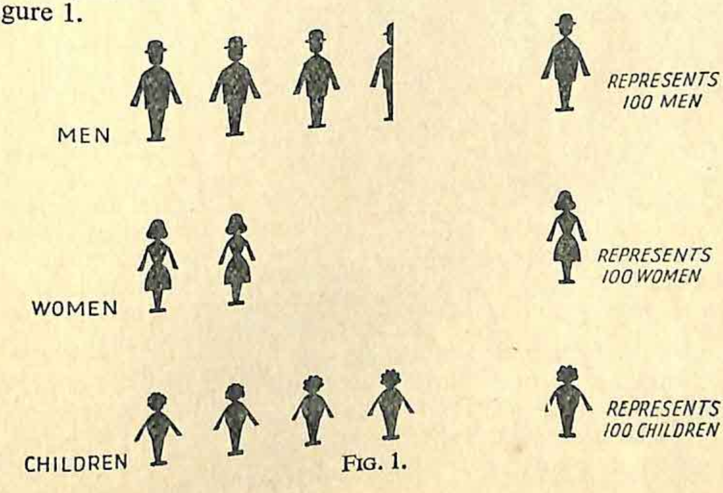


FIG. 1.

Any fraction of 100 less than $\frac{1}{2}$ would be difficult to represent. It is wise to avoid this problem either by choosing a different unit, say 50 men, or working in round numbers. This figure could equally well serve to illustrate 3,500 men, 2,000 women and 4,000 children, if each picture represented ten times as many people. The figure can be used to make comparisons of two kinds: differences and ratios. The statements which could be made include

(*a*) There are fewer men than children.
(*b*) There are 150 more men than women.
(*c*) There are twice as many children as women.
(*d*) There are seven men to every four women.

(*a*) and (*b*) are statements of difference and will readily be volunteered; (*c*) and (*d*) involve ratio and would be equally true if each picture represented a thousand persons, or fifty. The ratio statements are the more difficult and (*d*) in particular will only be volunteered after much practice with simpler cases.

When the numbers to be represented have more significant figures than those shown in Figure 1 above it may be necessary to work to, say, *two* significant figures. This method gives most useful insight into the meaning and use of approximation and of such a phrase as *to the nearest thousand.*

### 6·22. Calculation with Small Integral Numbers ; Tables.

Various ways of consolidating the work of the Primary School on applying the four rules to small numbers are suggested in the section on remedial work (4·4). It is likely however that some of the useful relationships between small numbers will be unfamiliar to most pupils. For instance, although the multiplication tables will commonly have been well learned, the idea of factors will often have been neglected. Measures may have introduced the idea of continued products e.g. 7 yards $= 7 \times 3 \times 12$ inches but little experience may have been given of writing 252 in its factor form $7 \times 3 \times 12$. Nor may the pupil have thought of 7 and 36 as associated factors which make 252, nor have learnt to find the factor to pair with 7 by dividing 252 by 7. There is a tendency to teach the multiplication tables in the form $7 \times 9 = 63$ without giving sufficient attention to the reverse form $63 = 7 \times ?$.

Computation can be greatly eased if numbers can readily be split into small prime factors. It is a good plan to practise in writing some frequently used numbers in their factor form. For example, $240 = 20 \times 12 = (5 \times 2 \times 2) \times (2 \times 2 \times 3)$. This can then be re-written as $2 \times 2 \times 2 \times 2 \times 3 \times 5$ in order to draw attention to the pattern of the factors and to lead to finding other pairs of factors which will make 240. The value of such knowledge about the structure of certain numbers will be seen later in dealing with fractions as well as with money and measures.

**6·221.** It must be assumed in what follows about written computation that the pupils are already familiar with the addition–subtraction and multiplication–division tables and such extensions as will have been learnt through money and measures. Through use these tables will become progressively better and more significantly known and therefore more rapidly and certainly on call. To secure speedy and confident calculation, flexibility in dealing with combinations of numbers must be extended throughout secondary school life and well beyond the range required for the correct performance of the written procedures now to be discussed.

### 6·23. The Four Rules for Integers Greater than Ten.

It is suggested in 2·41 (*c*) that many modern school pupils require some remedial work in the four rules for numbers greater than ten. In 4·812 an example is given of a fresh approach to the rule for multiplication; through new work in area-finding the structure of this rule is re-taught. Furthermore a plea is made for a gradual re-teaching of the elements of which the complex rule is made up rather than a re-teaching of the procedure as a whole. The importance of a thorough grasp of the four rules must be stressed for, with their constituent tables, they are the only fundamentals of all computation. When the idea of number is extended beyond positive integers, for example to fractions, the meaning of the four processes, addition, subtraction, multiplication and division must be correspondingly extended but the new computations are still based on the fundamental rules. Similarly when calculations relate to money or measures the same rules operate but must deal with arbitrary groupings of threes, twelves, fourteens, etc. instead of tens.

One difficulty deserves special consideration at this point. Although most pupils entering the Modern School will have been introduced to some, if not all, of the ' long ' rules, it does not follow that all will use the same pattern for working and setting down. Teachers in many areas have spent a great deal of time discussing this problem and where agreement has been reached on the methods to be used it is undoubtedly a great convenience. However, if a pupil is secure in a pattern of working which is not that of the majority, it is unwise to force a change of habit, particularly in the *mental working* involved. Some changes in the way the work is *set down* can be achieved easily, provided that the pattern has been understood and not accepted unthinkingly. For example, in multiplying by 23, a pupil who multiplies first by the 3 and then by the 20 may not find change of order difficult if he is clear that it is multiplication by 20 and not by 2 that he is performing and that this determines the position of the figures of the partial product. The stage by stage remedial work suggested in 4·812 may well be used to establish new patterns of working without the pupil experiencing any sense of change.

Some discussion of the ways rules are taught and the patterns of setting them down most often favoured is given in the Association's *Mathematics in Primary Schools*.

**6·231.** Any rule which has not been taught at the primary stage, should not be introduced in the secondary school as just another rule for reckoning. Some real need for it should first be seen and the approach to it by numerical stages should follow from that need. For example, long division will often have to be taught as a new skill. It is not difficult to find circumstances in which division by 2 and by 20 would arise before division in similar circumstances by 23. For example, suppose seating has to be arranged in rows of 23. How many rows will be required to seat 300? If the number in each row had been 20, children could experiment and discover for themselves some method of finding the answer, possibly by arranging double rows of 10. Such a method cannot be used for a prime number like 23. If the pupils are left to their own way of reaching an answer they are likely to make a count of twenty-threes: 23, 46, 69, 92, etc. When they arrive at ten twenty-threes, (230), they may realise that there are still 70 people to seat, and may see that 3 more rows would seat all but 1 of them. The teacher now has the opportunity to show that in such a procedure lies a method of dealing with all divisions and can set it out in the usual form.

| | |
|---|---|
| 1 | 23 |
| 2 | 46 |
| 3 | 69 |
| 4 | 92 |
| etc. | |

$$\begin{array}{r} 13 \text{ rows} \\ 23\overline{)300} \\ \underline{230} \\ 70 \\ \underline{69} \\ 1 \text{ Rem.} \end{array}$$

*Note*
*At later stages*
*the* 0 *of* 230
*is omitted.*

The correct answer to the original question must be discussed. Is it '13 rows with 1 extra seat'? Or should it be 'There must be 14 rows to make sure that everyone can be seated'? Ample practice in dealing with real situations of this kind will ensure that quotients and remainders are properly interpreted; we shall refer to this point again.

The actual procedure of long division will have to be systematised, and for many pupils the tabulating of products, i.e. the making of a table by continued addition as set out above, will offer the most reliable aid. Gradually the number of products to be written will be limited to those suggested by inspection, or by using an approximate divisor, such as 30 for 29, or 40 for 43, and will be found by multiplying.

Example:

$$\begin{array}{r} 28 \\ 29\overline{)827} \\ \underline{58} \\ 247 \\ \underline{232} \\ 15 \text{ Rem.} \end{array}$$

| | |
|---|---|
| 2 ..... | 58 |
| 3 ..... | 87 |
| 9 ..... | 261 |
| 8 ..... | 232 |

*Note:* Trial by 30 suggests 8, but this might be too small: try both 8 and 9 times. A pupil who has reached this stage, and understands what he is doing, should be able to check by dividing 232 by 8 mentally before using this product in the example.

It has to be remembered that long division is a complicated process, frequently involving subtraction, multiplication and addition, and the possibilities of error are many. Even the building up of the table by addition may contain a slip and it is wise to insist on careful checks: for instance the table can be carried on to 10 times when any error would become apparent, or for isolated products a check by division is worth while as suggested in the example above. At every stage pupils should be on the alert to notice if a remainder is bigger than the divisor. Confidence in dealing with 2-figure divisors is worth securing by frequent practice of a method on which pupils feel they can rely, but struggling to work with large divisors is not necessary for the demands of ordinary life. Indeed, skill in performing long division is of little value unless the correct form of answer to a particular question can be given, and this depends on the circumstances.

If you have 14 bricks you can take 3 bricks from 14 bricks 4 *times* and there are two *bricks* over; you share 14 bricks among 3 boys and you can either give them unequal shares or give them 4 *bricks* each with two *bricks* over. On the other hand the answer to the question ' how long is each piece if 14 inches of tape is cut into 3 equal parts? ' is $4\frac{2}{3}$ inches. Which then is the answer when 14 is divided by 3? Is it 4 and *two over* or 4 and *two-thirds*. The pupil must become used to giving his answer in the form appropriate to circumstances.

**6·232.** In general the size of numbers used in any type of computation should not be greatly increased just for the sake of practice. The rule should be adapted to larger numbers when they occur in the problems under consideration. This does not mean that a child who wants to show off his skill as a computer should not be allowed to do so, but computation should not be made burdensome for the majority. It is far better to introduce special calculating devices such as the slide rule or logarithm tables (6·25) than to hold up more significant work by labouring computation.

### 6·24. Fractions and Calculation with Fractional Numbers.

Many children entering Modern Schools have little knowledge of fractions except a practical acquaintance with a few simple fractions in particular contexts. Halves, thirds, quarters are known by name and can normally be accurately used.

In a teaching sequence the correct naming and writing of a few fractions occurring in particular experiences forms the groundwork, e.g. a half-penny, $\frac{1}{2}$d.; a quarter of a pound, $\frac{1}{4}$ lb.; three quarters of an hour, $\frac{3}{4}$ hr.; one third of a pint, $\frac{1}{3}$ pt.; and so on. This groundwork is then extended to the naming of fractions beyond the quarters such as fifths, one-sixth, three-sevenths. At this point ample practical experience must be given to ensure that pupils understand that the parts into which the whole has been divided must be *equal*. The next stage is to make clear the functions of the numbers which form the top and bottom of the fraction. For these purposes, and many others, diagrammatic ' wholes ' divided into equal parts usefully take the place of the objects and measures which are the starting-point of work with fractions.

**6·241. Use of explanatory diagrams.** Various forms of diagrammatic explanation of fractional notation are used by teachers. These should at first be associated with familiar uses of fractions and represent a unit, and its parts, of weight, time, money, etc. Otherwise there is a danger that the diagram may, as it were, *become* the fraction rather than a generalisation of situations involving the fraction. A variety of diagrams is in itself a safeguard.

Diagrams with the whole represented by an area, in particular a circular area, are common. There is much interesting geometry involved in the folding or dividing of shapes into equal parts in different ways, e.g. in realising that a square may be quartered by its diagonals or by lines through the mid points of opposite sides giving triangular quarters in one case and square quarters in the other.

The perfect symmetry of the circle makes it a specially appropriate figure for division. When compasses can be used to divide the circumference into sixths a circle can most conveniently show the properties of the twelfths group. Later when angular measurement is known, the dividing up of 360° at the centre makes the circular diagram valuable for displaying statistics giving fractional parts of a whole. If the circular area is used to represent a 24 hour day in a pupil's life, sectors can be drawn to show the number of hours he spends in different activities, e.g. sleeping, eating, playing and working in school. A circle shows quantities up to a single whole very clearly, but it is limited to this, whereas segments measured along a line can be continued to represent fractions of any size.

This mapping of fractions along a line such as the ruler's edge is for theoretical reasons the standard form of representation, but a long rectangle like the rule itself serves most early purposes. Thus a piece of card 12 inches by 1 inch might serve conveniently as *one whole* when considering the *twelfths family*. Pieces of the same width, say, twelve 1″ long, six 2″ long, four 3″ long, and three 4″ long, would make it possible to experiment with 12ths., 6ths., 4ths., and 3rds. The superposition of transparent diagrams of fractional parts can also usefully demonstrate the meaning of a written fraction. A variety of diagrams and material is of great value. For example, to demonstrate that $\frac{3}{12}$ is the same as one quarter, three pennies, 3 of 12, can be equated to a threepenny piece, 1 of 4, and expressed as a fraction of a shilling in two ways; or a ruler marked in 12ths of inches will show clearly the three spaces in each $\frac{1}{4}$ inch; or the 3 twelfths may cover the quarter in the apparatus described above.

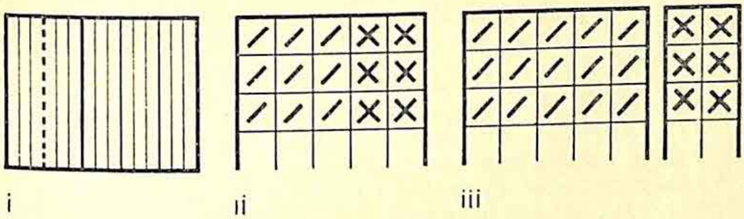
So far the notation of fractions has been used to express the relationship of a part to a whole, but it has other uses such as to express the ratio of two separate magnitudes, and diagrams also help to explain these. Again to possess $1\frac{1}{2}$ buns for oneself is a different situation from having 12 buns and sharing them equally amongst 8 people, though the result of doing this is to give each person $1\frac{1}{2}$ buns. The difference of the situations can be shown diagrammatically and equivalence of the numbers in the results justifies writing $12 \div 8$ as $\frac{12}{8}$ and then treating $\frac{12}{8}$ as a fraction.

**6·242.** Before fractions are established as a system to which the word 'number' can conveniently be extended, the pupil must be familiar with all forms of fractional numbers, i.e. he must be at ease with improper fractions, and with wholes expressed in such fractional forms as

$$\frac{8}{8},\ \frac{1}{1},\ \frac{8}{4},\ \frac{2}{1},\ \text{etc.}$$

**6·243. The principle of equivalence.** The equivalence of such a series of fractions as $\frac{3}{4}, \frac{9}{12}, \frac{15}{20}, \frac{75}{100}, \frac{7\frac{1}{2}}{10}$, must become familiar through many experiences and varied diagrammatic treatment.

Such questions as 'two out of every five; how many out of 15?' will bring out the numerical relationships. Coloured squares in a series of columns will help to make the generalisation clear, (Fig. 2). A pair of equivalent fractions gives four numbers in *proportion* (6·183 *c*).



$\frac{2}{5} = \frac{6}{15}$ 2 OUT OF EVERY 5 IS 6 OUT OF EVERY 15 2 FOR EVERY 5 IS 6 FOR EVERY 15

FIG. 2.

When the numerical relationship has been properly understood it can be used confidently either for reducing a fraction or for raising its denominator. It is important to stress the idea of a common factor or a common multiplier in a numerator and denominator in the early stages rather than to speak of ' cancelling '.

The principle of equivalence is sufficient for solving any problem involving fractions which a pupil is likely to meet in experience and it is well to give time for such solutions by first principles before discussing ' the four rules applied to fractions '.

*Example* 1. What is the total length required if 5 girls want $\frac{1}{2}$ yard each and 3 girls only $\frac{3}{8}$ yard each?
This could be worked as follows:
$\frac{5}{2}$ yd. and $\frac{3}{8}$ yd. $\times 3$
$= 2\frac{1}{2}$ yd. and $\frac{9}{8}$ yd.
$= 2\frac{4}{8}$ yd. and $1\frac{1}{8}$ yd.
$= 3\frac{5}{8}$ yd.

*Example* 2. What is $\frac{1}{3}$ of $6\frac{1}{2}$ yards? (or share $6\frac{1}{2}$ yd. among 3 girls)
$\frac{1}{3}$ of 6 yd. $= 2$ yd.
$\frac{1}{3}$ of $\frac{1}{2}$ yd. $= \frac{1}{3}$ of $\frac{3}{6}$ yd. (changing $\frac{1}{2}$ into $\frac{3}{6}$ to make the numerator divisible by 3).
$= \frac{1}{6}$ yd.
Total $2\frac{1}{6}$ yd.

**6·244.** It is usual and convenient to take pupils of average ability to the stage of treating fractions as numbers and extending the meaning of addition, subtraction, multiplication and division so that the four rules can be applied generally in problems involving both whole numbers and fractions. It is not necessary to draw the pupil's attention to this extension but if the teacher is aware of what he is doing he will make the extension of a word such as ' multiplication ' applied to fractions appear as a normal process. An arbitrary re-definition however correct may only confuse the pupils at this stage.

Difficulty does not arise until the multiplication involves a fraction less than one. Then the pupil may complain ' But the answer is smaller, multiplication makes things bigger'. Perhaps the most convincing argument is to show that the smaller the multiplier, the less the result in such a series as

4 trays holding 24 eggs per tray hold $4 \times 24$ or 96 eggs.
3 trays holding 24 eggs per tray hold $3 \times 24$ or 72 eggs.
2 trays holding 24 eggs per tray hold $2 \times 24$ or 48 eggs.
$\frac{1}{2}$ tray holding 24 eggs per tray holds $\frac{1}{2} \times 24$ or 12 eggs.
$\frac{1}{3}$ tray holding 24 eggs per tray holds $\frac{1}{3} \times 24$ or 8 eggs.
$\frac{1}{4}$ tray holding 24 eggs per tray holds $\frac{1}{4} \times 24$ or 6 eggs.

This method implies that dividing by 2 or 3 may be written as multiplication by $\frac{1}{2}$ or $\frac{1}{3}$. The extension to multiplication by such a fraction as $\frac{2}{3}$ follows easily.

**6·245. Addition and subtraction of fractions.** It is recommended, as in the Association's *Mathematics in Primary Schools*, that addition and subtraction of fractions be done from first principles and that the bracket line for the common denominator be omitted altogether for many children. Abler pupils may formalise these operations at a later stage and generalise the results using literal symbols such as

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

Furthermore the use of the *lowest* common denominator is unnecessary; any convenient common denominator suffices until pupils themselves through their growing knowledge of number in factor form seek economical denominators. At first work should be confined to commonly useful families of fractions, twelfths, sixteenths, tenths and perhaps twenty-fourths. For many pupils this will be the limit of their useful achievements.

In subtraction the main difficulty occurs when a large fractional part must be taken from a smaller one. This is eased if the two fractional parts are not brought together over a single denominator until the subtraction is actually performed. Thus:

$$\begin{aligned} 4\tfrac{1}{3} - 1\tfrac{5}{6} &= 4\tfrac{2}{6} - 1\tfrac{5}{6} \\ &= 3\tfrac{8}{6} - 1\tfrac{5}{6} \\ &= 2\tfrac{3}{6} \\ &= 2\tfrac{1}{2}. \end{aligned}$$

Throughout the work in addition and subtraction of fractions, examples should be associated with measures or money, and in particular with lengths and areas. Diagrammatic representation and the use of scales constantly give rise to computation involving fractions; indeed it is possible that the poverty of geometrical experience usually offered to girls has been responsible for some of the difficulties they have encountered in working with fractions.

**6·246. Multiplication and division of fractions.** (See sections 9·5 and 9·6 of the Association's *Teaching of Arithmetic*.) Examples of multiplication and division in which only one element is a fraction have already been given. The 'multiplication' or 'division' of a fraction by a fraction is another matter. It may appear arbitrary simply to identify the multiplication sign with ' of ' in such a phrase as $\frac{2}{3}$ of $\frac{4}{5}$. Presented as a practical problem this can be found from first principles, using the meaning of the fraction of a quantity and the principle of equivalence, thus:

$$\begin{aligned} \tfrac{1}{3} \text{ of } \tfrac{4}{5} &= \tfrac{1}{3} \text{ of } \tfrac{12}{15} \\ &= \tfrac{4}{15} \\ \therefore\ \tfrac{2}{3} \text{ of } \tfrac{4}{5} &= \tfrac{8}{15}. \text{ But this is } \tfrac{2\times 4}{3\times 5} \end{aligned}$$

Further examples establish the pattern and suggest the reasonableness of writing $\frac{2}{3}\times\frac{4}{5}$ instead of writing ' of '.

Two diagrammatic ways of demonstrating the process can be suggested. They assume that the multiplication rule for finding the area of a rectangle is known.

To find $\frac{1}{2}\times\frac{3}{4}$

(1) A rectangle $\frac{1}{2}$ in. by $\frac{3}{4}$ in. is drawn within a square inch. (Fig. 3i.) The units are changed to $\frac{1}{4}$ inches and the square inch subdivided into $\frac{1}{4}$ inch squares of which there are 16 in the whole square inch. Thus the rectangle $\frac{1}{2}$ in. by $\frac{3}{4}$ in. is seen to contain six sixteenths or $\frac{3}{8}$ of a square inch.

(2) Or a rectangle $\frac{1}{2}$ in. by $\frac{3}{4}$ in. is drawn and is shown to contain 3 rectangles each $\frac{1}{8}$ square inch. (Fig. 3ii.)

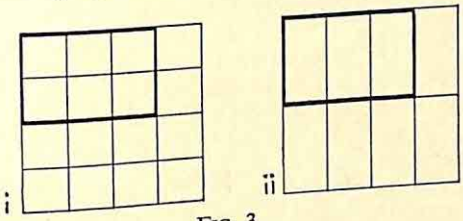


FIG. 3.

Other practical ways of demonstrating will suggest themselves to teachers. For example, a length of $\frac{3}{4}$ yd. of ticker tape can be halved by folding. The half is 13½″ long. Folding a yard shows that $\frac{1}{8}$ yd. is 4½″ long; 13½″ $=\frac{3}{8}$ yd. so half of $\frac{3}{4}$ yd. is $\frac{3}{8}$ yd. Money offers a good illustration because fractions of a shilling can be converted into pence: thus,

$$\begin{aligned} \tfrac{1}{2} \text{ of } 1\tfrac{2}{3} \text{ sh. or } \tfrac{1}{2} \text{ of } \tfrac{5}{3} \text{ sh. is } \tfrac{1}{2} \text{ of 1 sh. 8d.} &= 10\text{d.} \\ &= \tfrac{10}{12} \text{ sh. or } \tfrac{5}{6} \text{ sh.} \end{aligned}$$

Such methods lead to an easy acceptance of the use of the multiplication sign and the rule for multiplying when applied to fractions.

The idea of division by a fraction will probably follow readily from its relationship to multiplication.

e.g. If $\frac{1}{2}\times\frac{3}{4}=\frac{3}{8}$, then $\frac{3}{8}\div\frac{3}{4}=\frac{1}{2}$.

It can also be seen to have a meaning in terms of one fraction being contained in another.

e.g. $\frac{3}{8}$ contains $\frac{1}{8}$ three times i.e. $\frac{3}{8}\div\frac{1}{8}=3$
$\frac{3}{8}$ contains $\frac{1}{4}$ $1\frac{1}{2}$ times i.e. $\frac{3}{8}\div\frac{1}{4}=1\frac{1}{2}$.

This can be shown on a ruler or illustrated with money.

To find how many times $\frac{3}{8}$ contain $\frac{1}{4}$ most pupils would think of $\frac{3}{8}$ as $\frac{1}{4}+\frac{1}{8}$ or a quarter and its half, and give the answer as $1\frac{1}{2}$ at once.

$\frac{4}{9} \div \frac{2}{3}$ is a more complicated process but can be dealt with on similar lines:

$$\frac{4}{9}\div\frac{6}{9}=4\div 6=\frac{2}{3}.$$

The general rule is seen when such fractions as $\frac{4}{5}$ and $\frac{2}{3}$ are treated similarly

$$\begin{aligned}\frac{4}{5}\div\frac{2}{3}&=\frac{4\times 3}{15}\div\frac{2\times 5}{15}\\&=(4\times 3)\div(2\times 5)\\&=\frac{4\times 3}{2\times 5}\text{ which can be written as }\frac{4}{5}\times\frac{3}{2}\end{aligned}$$

Thus the rule for inverting the divisor and multiplying will be accepted by the working of several examples.

**6·247. Decimal fractions.** In Britain, because of our system of measures, it seems harder to induce pupils to think decimally than to make them familiar with the properties of vulgar fractions. Yet advertisements frequently quote decimals; e.g. one firm advertised the thickness of the oil film in a car as ·002″. Decimal numbers to show the degree of accuracy of a machine are commonplace. A class which collects examples of such usages can have a most stimulating introduction to the importance of decimals. The development of technical processes and the use of calculating machines will make familiarity with the notation of decimal fractions essential for both producers and consumers of goods. The power to use such fractions readily when appropriate must precede any attempt to learn 'the four rules applied to decimals'. Fortunately scale drawing and graphical work give good opportunities within the range of the pupils' own activities for stating measures 'correct to the nearest tenth' of a unit, or even for estimating tenths of a tenth. Further they will give rise to simple computation which can often be checked by actual measurement. Later, practical topics such as surveying will provide a full experience of calculating in decimals.

There are several uses of decimal fractions which need to be brought out:

(*a*) Whenever quantities are measured in units related to one another in tens, e.g.

100 links = 1 chain.
1000 metres = 1 kilometre.
10 florins = £1.

the decimal notation is economical and clear. Thus 7 chains 65 links can be written 7·65 chains and recognised at once as 765 links, since the second decimal place represents hundredths, or links.

(*b*) Because our number system is based on tens it has been customary for purposes of comparison to use *per cent* or *per thousand*. In schools much of the calculation of percentages has been taught in terms of vulgar fractions, e.g.

$$12\tfrac{1}{2}\% = \tfrac{25}{200} = \tfrac{1}{8};\quad 8\% = \tfrac{8}{100} = \tfrac{2}{25};$$

whereas expressed as decimals, 8% = ·08; $12\frac{1}{2}$% = ·125.

Conversely, ·075 can be read at once as 7½%. If, as seems possible, our coinage is modified into a decimal system within the next few years most of our computations will be by means of decimals.

(*c*) It is possible through the use of decimal fractions to work to any required degree of accuracy consistent with the reliability of the data (6·85). The number of significant figures can be seen as readily for small quantities as for large. For example, ·006375 can be given as ·0064 to 2 significant figures, just as 95,673 can be stated as 96,000 also to 2 significant figures. The standard notations for very large and very small numbers illustrate this use of decimal notation. Such a number as 78,452,314 may be accurate to only 3 significant figures. It could be expressed as 78,500,000 or 78½ millions; in standard form the first figure becomes the unit digit and the resultant number 7·85 is multiplied by $10^7$ to give its true value, i.e. it is written $7{\cdot}85 \times 10^7$. A small number such as ·0003742 given in standard form to 3 significant figures would be $3{\cdot}74 \times 10^{-4}$. This combination of powers of ten with a standard decimal form is so widely used that it is well worth while to build up an understanding of it. Some preliminary work on negative powers of ten will however be necessary. The various ways of expressing decimal fractions are illustrated in the statement that the average width of a hair is given in inches as ·0025, or 2½ 'thous', or $2{\cdot}5 \times 10^{-3}$. Practice in writing and interpreting such decimal measures is more fruitful than much mechanical computation.

**6·248. The four rules applied to decimals.** (Association's *Teaching of Arithmetic*, section 10.) When the use and meaning of decimal notation have been understood, little difficulty is experienced in addition and subtraction, particularly if applied to measured lengths. Just as feet and inches are arranged in separate columns to be added, so whole inches will be arranged in proper columns with the decimal parts on the right, e.g.

$$\begin{array}{r@{\cdot}l} 3 & 2 \\ 15 & 05 \\ & 6 \\ \hline \end{array}$$

Similarly multiplication by a whole number will be readily accepted, e.g. 17 times 3·5 can be read as (17 × 35) tenths. The multiplication can be carried out as a whole-number operation and the point inserted in the answer at the end. It is a different matter if the multiplier also includes a decimal fraction. Such a multiplication could arise in finding the area of a rectangle 1·2 in. by ·3 in. The drawing of a diagram would lead to the realisation that the small squares are hundredths of a square inch and the correct answer is 36 hundredths or ·36 sq. in. (Fig. 4.)

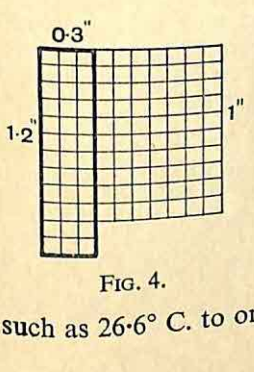


FIG. 4.

Alternatively, multiplication might be needed in changing from British to metric units or in converting a temperature expressed in degrees Centigrade such as 26·6° C. to one in degrees Fahrenheit.

*Example:*

$$1^\circ\text{ C.} = 1{\cdot}8^\circ\text{ F.}$$
$$26{\cdot}6^\circ\text{ C.} = (26{\cdot}6 \times 1{\cdot}8)\text{ Fahrenheit degrees.}$$

If this is read as 266 tenths × 18 tenths, the answer is seen to be hundredths.

$$\begin{array}{r} 266 \times 18 \\ 18 \\ \hline 266 \\ 2128 \\ \hline 4788 \end{array}$$

But this is hundredths.

∴ 47·88 is the equivalent in Fahrenheit degrees.

The Fahrenheit temperature is then found by adding 32°.

It is recommended that when abler pupils meet a situation like one of the foregoing in which decimals need to be multiplied, the rule established should be the simple one of multiplying as for whole numbers and finding the place of the point in the product by counting places in the multiplier and multiplicand.

If division is attempted it is well for pupils to be familiar already with the 'fraction bar' way of writing division, i.e. $\frac{85}{16}$ for 85 ÷ 16. It is then easy to write 427·63 ÷ 13·5 as $\dfrac{427{\cdot}63}{13{\cdot}5}$

and to re-write it in tenths as $\dfrac{4276{\cdot}3}{135}$

or in tens as $\dfrac{42{\cdot}763}{1{\cdot}35}$

In either case the position of the point in the quotient is readily seen.

The use of the fraction bar also makes it easy to convert a vulgar fraction, such as $\frac{4}{7}$ into a decimal. This is a valuable technique since the decimal can be found to as many places as are required for the particular problem to be solved.

**6·249. Percentages.** Paragraphs 6·151 and 6·247 both refer to the introduction of percentages as part of the teaching of fractions. Besides being able to recognise 16% as ·16, or as a vulgar fraction, pupils need to be able to find 16% of, say, 452 or 3067 and this can be done directly by finding the number of hundreds, 4·52 or 30·67, and multiplying by 16. The converse problem of finding what percentage 85 is of 290 will probably be done more easily as $\frac{85}{290}$ of $100 = \frac{8500}{290}$, which can be worked as a division of decimals:

$$\begin{array}{r} 29{\cdot}3\text{[illegible]} \\ \hline 29)850\phantom{{\cdot}0} \\ 58\phantom{{\cdot}0} \\ \hline 270\phantom{{\cdot}0} \\ 261\phantom{{\cdot}0} \\ \hline 9{\cdot}0 \\ 8{\cdot}7 \\ \hline \end{array}$$

It is also important to practise recognising that $53{\cdot}2\% + 46{\cdot}8\% = 100\%$ so that if 53·2% of the electorate went to the polls 46·8% did not.

Increase (and decrease) per cent should also be studied. If a group increases from 40 to 50 that is an increase of 25%, but the newcomers form only 20% of the final group. Another idea which proves puzzling is that if 100 is increased to 120 and then there is another increase of 20% the final number is 144.

### 6·25. Other Methods of Computation.

Boys and girls who would like to carry their mathematical studies into some of the interesting fields open to them may be discouraged if the computation is too burdensome. There is no reason why they should not use all available aids to lighten the burden. Such aids play the same part in mathematics as the sewing machine does in dressmaking. It is essential that the operator should know how to thread and work the sewing machine but it is not necessary to understand the subtle mechanism of the shuttle and the interlocking threads. Similarly calculating machines and tabulated results may be used intelligently even though their construction remains a mystery.

The *making* and using of *simple* tabulated ready reckoners is suggested as one means of remedying weakness in the rules of reckoning. Graphical ready reckoners are discussed in 6·48 (*c*). Tables of squares and of reciprocals can also be constructed in outline to serve some special needs, and can then be supplemented by the fuller published tables. The ability to use such published tables is worth acquiring and, though the detailed arrangements of tables vary according to their nature, familiarity with one type is a help in mastering others.

**6·251. Logarithms : the case for introducing them.** (See the Association's *Teaching of Arithmetic* section 14 and *Teaching of Algebra* section 8.) Multiplication has always been recognised as involving a heavy task of memorising, and for centuries men have tried to devise ways of avoiding the labour—Napier's Bones (to be seen in the Science Museum in London) provided one ingenious way of literally carrying the multiplication tables with one and using diagonal addition to give each partial product. But they are cumbersome and the invention of logarithms by Napier gives him a greater claim to fame.

Pupils of quite modest ability can learn to use logarithms for multiplication and division. It is not necessary to give any detailed explanation of the rules for using them, or their practical version, the slide rule, before multiplication is carried out with their aid. Detailed suggestions for introducing logarithms and making a slide rule are given in 6·28. Here it need only be emphasised that many pupils feel a new respect for Mathematics when they realise the labour-saving properties of logarithms and for this reason, even if their future use will be small, it is a valuable experience for a boy or girl to make their acquaintance.

**6·252. Square root.** The idea of a square root may occur in investigating the areas of squares on the sides of a right-angled triangle when the theorem of Pythagoras is being studied. For example, in the triangle which has sides of 1″ and 2″ including a right angle, the third side will have upon it a square of area 5 sq. in. Measurement of its length and calculating the area of the square gives a result which is not exactly 5. If the square numbers have already been graphed and a smooth curve drawn through the points that represent them, a number can be read which, squared, gives approximately 5, or any other chosen number.

Some pupils may feel dissatisfied if they are not assured that there is an arithmetical device for finding a square root to any required degree of accuracy, but it is suggested that it is not worth teaching the long arithmetical rule when logarithms and square root tables are readily available. The properties of square numbers and their differences are discussed in 6·211. The geometrical properties of squares are treated in 6·5.

### 6·26. Checks.

We have already made some suggestions that the pupil who is engaged in making a computation should check the steps of his working: here we make a few observations on the importance of checking a final result.

The use of checks on all possible occasions is sometimes advocated, but there is a danger that the zealous teacher may pursue this aim too far. It is worth while to spend time teaching the child to take all *reasonable* steps to test the accuracy or validity of a result, but the definition of ' reasonable ' will depend upon the individual child. It is clearly not fruitful to teach a method of checking which will involve more labour or difficulty than the original working; for example, it will not be fruitful as a rule to enjoin checking the accuracy of a long multiplication by performing the corresponding division, since the latter is more likely to give rise to computational error. Again, the testing of a multiplication by ' casting out nines ' may be an attractive example of arithmetical magic to some children—not necessarily the most able—but should not be considered applicable for all, although the computation is easy.

**6·261.** Checks fall into two main classes, practical and computational. The practical check consists in the pupil's satisfying himself of the reasonableness of his result on grounds of ' common sense ', as when he makes a rough estimate, in units he can readily visualise, of the cost of 5½ lb. of meat, or the area of a room or field. The growth in recent years of teaching methods which constantly have recourse to practical experience and experiment has fostered the habit of practical checks, and the teacher should actively encourage this; the ' common sense ' check on reasonableness does not, however, always lend itself to simple expression in writing, and the teacher will need to verify orally that it is being used in appropriate cases.

**6·262.** Accuracy in computation is fundamental and essential. The answer to a computation can be only right or wrong; there is no such thing as a degree of rightness, as ' nearly right '. Suppose, for example, that the result of a calculation should be 5615, and that A and B have got 6615 and 5616 respectively. There is a tendency for an inexperienced teacher to regard B's answer as better than A's, because it is ' only 1 out ' whereas A's is ' a thousand out '; this point of view is quite unsound, arising probably from the teacher's failure to distinguish between accuracy and reliability: mechanical learning of procedure in four-figure logarithms seems to result in the view that anything is permissible in the fourth figure *in any computation*. The two answers given above are both *wrong*, one is tempted to say 'equally wrong', but the qualifying word 'equally' is justifiable only if it is known that the answers arise from the same failure in table knowledge. Tables must be known, to be used with certainty; the only acceptable standard of accuracy is 100%. Standards of reliability will be discussed in a subsequent section (6·85).

**6·263.** In any computation it is worth while, failing an independent check, to go over the working again, preferably after a few minutes at least have been given to some other question or, better, some quite different activity: a short break in time between the two acts may reduce the risk of repeating a mistake. It is preferable, however, to use an independent check which will bring into operation different sets of number relations or processes. The most familiar is the practice of adding a column of figures both up and down, or a row of figures from both right and left. A development of this is the check afforded by the 'grand total' of tabulated items which have been added both vertically and horizontally: 'long tots and cross tots'; a table of numbers may be usefully set periodically as 'self-checking' practice in addition, as in the once popular 'Civil Service Tots'. Subtractions should be checked by additions. These are the only cases which the teacher may confidently regard as worthy of general application.

**6·264.** Multiplication and division do not lend themselves to any economical exact checks, but in decimal multiplication and division the fixing of the decimal point in an answer may usefully be checked by an alternative method. If in multiplication the decimal point is fixed by 'counting the decimal places' it may be checked by a rough estimate: e.g. 27·4 × 0·062 is roughly $30 \times \frac{6}{100} = \frac{180}{100} = 1{\cdot}8$; to work the rough estimate decimally as 30 × 0·06 would be less effective because it would offer the opportunity for the same kind of mistake as might occur in the computation proper.

In division the possible occurrence of a remainder complicates the issue, and an exact check by multiplication is not advisable, though a rough check may be worth while. It will be noted here that the exact numerical check is giving place to the rough estimate, which lies between the two kinds of check first mentioned.

**6·265.** The problem of deciding how far to press the habit of checking is analogous to deciding how far one should teach short methods. Some types of check which may be applied intelligently and effectively by children having good number sense might for less able children tend only to make confusion more confounded. For example, suppose a child has to work 59 × 24; probably most children should be able to apply the check that the answer must be an even number; fewer might be expected to realise without prompting that the answer must be divisible by 3. Some might check by working 60 × 24 and then subtracting 24. To take another example, 37 × 31 may be worked by using either number as multiplier, and probably most pupils may be brought to realise that to do both affords a check. It is true, however, that the pupil who will most readily see opportunities for checks of special and limited applicability is least in need of dependence upon them; for such a pupil the devising of special checks is valuable in developing mathematical insight but should not be forced as an unnecessary prop to his confidence in his own manipulative skill.

The general principle is that the value of a check diminishes as it becomes more complicated or cumbersome; apart from the simple, certain, checks for addition and subtraction, the teacher should demand no more from the child than that he keep an eye on the reasonableness of his results.

### 6·27. Approximate Calculation.

When a pupil works 49 × 89 he knows that there is a definite right answer to be obtained and that he can complete his calculation. If he decides to work instead 50 × 90 that too has a correct answer, a rather larger one, and more easily obtained, but certainly not the same as the previous one. We have discussed in 6·231 whether the correct answer to 14 ÷ 3 is 4 and 2 over, or $4\frac{2}{3}$, and decided that it depends on the circumstances, but when the circumstances are known one of these is right, and the other is wrong: the only acceptable standard of accuracy is 100%.

When the idea of what a number is, has been extended to include decimal fractions, in working a division which does not 'come out' in integers a pupil may 'add noughts' in order to continue dividing as far as he wishes, or is instructed, to go. The answer he must obtain is no longer absolute but only 'correct to the nearest hundredth' or 'to three significant figures'. Unless great care is taken at this stage confidence in the certainty of Mathematics may be lost when an answer has no longer to be *exactly* right, but only approximately so. Results read from tables, or from graphs, suffer from this defect, and methods learned for checking results break down. Thus if $\frac{100}{23}$ has been found by division or from tables to be 4·348 and then 4·348 × 23 is worked the result is not exactly 100 but 100·004. The discrepancy is small, but enough to puzzle a conscientious pupil accustomed to *absolute* accuracy. Further difficulties arise when calculations are based on measurements but these will be discussed in section 6·85. Here we only draw attention to the approximate nature of work with logarithms.

### 6·28. Logarithms and the Slide Rule.

In order to understand how it is that by adding and subtracting logarithms we can find the results of multiplying and dividing numbers, it is necessary to be familiar with the laws of indices. But to be able to use logarithm tables intelligently no more is needed than a grasp of the index way of writing numbers using powers of ten set out in 6·247 (*c*).

There are several good ways of introducing logarithms; for pupils of limited mathematical ability the following is probably the best method. First set out powers of ten, as follows:

$10 = 10^1$
$100 = 10^2$
$1{,}000 = 10^3$
$10{,}000 = 10^4$
etc.

$100 \times 1{,}000 = 10^2 \times 10^3$
$100 \times 1{,}000 = 100{,}000$, which is $10^5$ or $10^{2+3}$
Similarly $10 \times 100{,}000 = 10^1 \times 10^5$
$10 \times 100{,}000 = 1{,}000{,}000 = 10^6$ or $10^{1+5}$

Mathematicians have been able to calculate decimal indices, or logarithms, for numbers lying between the powers of 10 and have tabulated them for our use. For example, they tell us that to 3 dec. places $2 = 10^{\cdot 301}$ and $3 = 10^{\cdot 477}$.

Add these indices

$$\begin{array}{r} \cdot 301 \\ \underline{\cdot 477} \\ \cdot 778 \end{array}$$

The tables tell us that $10^{\cdot 778}$ is 6.

Further simple examples will show how the tables work. It is strongly urged that from the start calculations with logarithms should be set out without equal signs and in a precise form such as the following—

| Number | Logarithm |
|---|---|
| 1·5 | ·1761 |
| 2·3 | ·3617 |
| 3·45 | ·5378 |

The logarithms of numbers greater than 10 involve finding the characteristic, or whole number part, of the logarithm. This can be done by referring to the whole number indices of the powers of ten set out above. This is a frame of reference which the pupils will find useful.

Thus 3256 is greater than 1000 and less than 10,000. Therefore its logarithm will be 3 plus a decimal part.

32·56 has 1 as *characteristic* of its logarithm.

3·256 has 0 as characteristic, and the logarithm is read directly from the tables.

The main stages to be observed are:

(*a*) Multiplication by numbers greater than 1.
(*b*) Division with dividend and divisor greater than 1, yielding a quotient greater than 1.
(*c*) Division by 10 or a power of 10 introducing a negative characteristic.
(*d*) General use of logarithms in evaluating products.

*Note.* Many pupils may find stage (*c*) too difficult. An alternative procedure is as follows:

(i) $4{\cdot}73 \times {\cdot}0825 = \dfrac{4{\cdot}73 \times 8{\cdot}25}{100}$

Calculate the numerator by logarithms.
Divide it by 100.

(ii) $\dfrac{16{\cdot}58}{{\cdot}0039} \quad = \dfrac{16{\cdot}58 \times 1000}{3{\cdot}9}$

Evaluate $\dfrac{16{\cdot}58}{3{\cdot}9}$ by logarithms and multiply the result by 1000. (The denominator must be less than the numerator.)

The evaluation of powers of a number should at first be treated from first principles as repeated multiplication through repeated addition of the logarithm of the factor involved. Later a rule may emerge in, for example, finding squares and square roots required in mensuration by doubling and halving the logarithms of the numbers. Problems of growth in which a growth factor is of significance can then be treated by logarithms (1·24).

Doubling the logarithm to find the square presents no difficulty but halving to find the square root needs care in the case of numbers less than 1.

*Example* $\sqrt{\cdot 359}$

If this is written as $\sqrt{\frac{3\cdot 59}{10}}$ in order to avoid a negative characteristic the pupil will not be able to deal with the denominator. It must be written as $\sqrt{\frac{35\cdot 9}{100}}$ in order that the square root of the denominator can be found by inspection, and the number written as $\frac{\sqrt{35\cdot 9}}{10}$. The numerator is then found by logarithms and divided by 10 to give the correct answer.

**6·281.** The use of a slide rule is fascinating to most young people and is readily seen to be a linear alternative to using logarithm tables. It is well to have slide rules available if possible so that pupils may use them in turn.

The making of a slide rule is an achievement coveted by many boys and will appeal to a fairly able class. Semi-logarithm paper can be bought and cut into convenient strips with the logarithm scale along their lengths. If two strips are stiffened and numbers are inserted along opposite logarithmic edges, the simplest form of slide rule can be demonstrated. Refinements such as fixing grooved slides can be added if time and opportunity allow.

### 6·29. Computation with Directed Numbers.

The meaning given to directed, or signed, numbers in interpreting experience is discussed in 6·152. The representation of these numbers by points along a line from a zero point in a chosen positive direction, generally left to right, and in the opposite negative direction, is a basis for the extension of addition and subtraction to them. Thus addition is achieved by successive steps along the line, and if such a step is itself negative, e.g. the second step in $(+5)+(-7)$ the effect of it is a subtraction of the magnitudes, giving in the example $+5\ -7$ or a resultant step of $(-2)$. Again $(-5)+(-7)$ are all steps in the negative direction or a resultant of $(-12)$. Similarly subtraction is achieved by comparison by difference of positions on the line in favour of the first-named number.
Thus:

$(+5)-(+2)=(+3)$, that is the first position is 3 steps to the right of the second

$(+5)-(+7)=(-2)$, that is the first position is 2 steps to the left of the second

$(+5)-(-7)=(+12)$, that is the first position is 12 steps to the right of the second

$(-7)-(+5)=(-12)$, that is the first position is 12 steps to the left of the second

and so on.

After some practice with a diagram and its interpretation, an activity possible at an early age, pupils may be able to formulate a 'rule of signs' which enables them to dispense with the diagram and the reasoning which accompanies it.

A meaning for multiplication and division of directed numbers involving a similar rule of signs is not so easily dealt with. Discussion of it may be based on the meaning of results when a formula involving a product is extended to cover signed numbers. This is somewhat outside the scope of this report, though with patient preparation it may be within the scope of ordinary pupils. Too often the step is taken arbitrarily in order to facilitate the teaching of formal algebra. The line taken throughout this report has been that expanding ideas of number follow closely the opportunities which give them and the processes involving them real significance. Such occasions for the multiplication of directed numbers may occur in some courses.

Some of the work in statistics (7·513) is made easier if pupils are prepared to use signed numbers for excess or defect from a mean.

## 6·3. Algebra.

*6·31. Uses of letters.*
*6·32. Extension of the idea of number, including directed number.*
*6·33. Index notation.*
*6·34. Algebraic form.*
*6·35. Equations.*
*6·36. Algebra used in other branches of Mathematics.*
*6·37. Algebra as a vocational study.*
*6·38. Additional notes on written symbols.*

This abstract branch of Mathematics, concerned with the generalisation and extension of arithmetic and the manipulation of symbols, is unlikely in itself to interest modern school pupils, nor is it probable that it will arise naturally out of the topics we have suggested. It should not be included as a separate subject of study in the curriculum, but there are certain considerations about the use of letters and other symbols which are placed together here for the convenience of the teacher. Though the pupils will not study algebra for its own sake some treatment of symbolism will be a necessary preliminary for the many who later require formulae etc. vocationally.

### 6·31. Uses of Letters.

Letters of the alphabet are frequently used to stand for numbers; adults interpret these letters in any particular case according to the context. Here we attempt to consider in what different circumstances such letters may be used.

**6·311.** There are letters which are used for specific numbers. Thus in the Roman system we have X for ten, V for five, L for fifty, C for a hundred. Again, in listing the points of an argument $a, b, c \ldots$ etc. are often used as alternatives to numerals. Then, too, certain important irrational numbers, notably $\pi$ and $e$, are always denoted by letters.

**6·312.** For many purposes it is traditional to use certain letters though these do not stand always for the same number as in the preceding paragraph; thus an integer is usually denoted by $n$, a coefficient of friction by $\mu$, an eccentricity by $e$, the first term of a series by $a$, the acceleration due to gravity by $g$. In this last case the same letter $g$ is used whether the units are centimetres per second per second, or feet per second per second, but the number is not the same in the two cases.

**6·313. Letters in formulae.** Pupils in school will be likely to use letters to formulate rules about what to do with measurements: ' Multiply the number of feet in the length by the number of feet in the breadth to find the area of a rectangle in square feet '; ' Divide the number of miles by the number of hours to find the speed '; ' Find the average length of the parallel sides and multiply by the height to find the area of a trapezium '; ' Subtract the principal from the final amount to find the compound interest '. For these, and other formulae, they can choose their own letters and should do so, though most textbooks use the same ones.

Some attention must be paid to seeing that the letters stand only for numbers and care is necessary to get the units right. Though letters are often used in formulae to stand for *quantities*, the practice of making letters stand for *numbers only* is strongly recommended for beginners. Such statements as ' distance over speed equals time ' cannot be properly understood without a mature and intimate knowledge of units and dimensions; to say that ' the number of miles divided by the number of miles per hour gives the number of hours ' reduces the matter to simple common sense. Similarly an unknown distance should not be called $x$, but, say, $x$ miles or $x$ cm.

Some formulae when found are useful as an aid to memorising a result. Thus after a long investigation and some theoretical thinking pupils discover that the area of a circle of radius $r$ can be calculated from the formula $A = \pi r^2$ and that the details of the method by which the formula was found need not be repeated or remembered; moreover the formula itself has an expected form, $r^2$ because it is for an area, $\pi$ because it is for a circle. By contrast, pupils find the simple interest in many numerical cases, and then find the interest on £$P$ in $t$ years at $r$% per annum by exactly the same method: they may remember this result, but they need not, and indeed should not, be required to memorise that $I = \frac{P \times r \times t}{100}$ or $\frac{Prt}{100}$: the formula, if not remembered, can be found again by repeating the train of reasoning by which it was found.

But all such formulae, whether remembered or just found, or even supplied by the teacher or a book, will be understood merely as mnemonics. It is an exercise to state a rule in this form: afterwards when the formula is used, specific numbers are substituted for the letters, and a numerical answer is obtained; there is no question of manipulating the formula or working with the letters.

**6·314. Letters in general statements.** It is rather different when letters are used to express mathematical truths about numbers. A pupil may know very well that $5 \times 7 = 7 \times 5$, but find it difficult to realise this as a particular case of $a \times b = b \times a$. He may learn to clear away the brackets from $p(q+r)$ without understanding that this too is a 'rule', and time could well be spent on illustrating it both numerically and graphically (Fig. 5). An opportunity may occur in problems on painting the walls of a room to obtain a similar formula $2h(l+b)$.

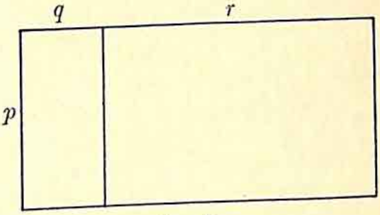


FIG. 5.

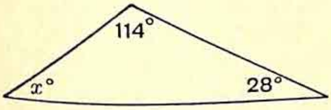
The converse exercise of inserting brackets after 'spotting' a common factor of two or more numbers will also help. Such expressions as $125+85=5(25+17)$; $64+36=4(16+9)$ lead to $7a+14b=7(a+2b)$ and later to examples which are all letters. Though not many modern school pupils will reach so far, teachers may realise the value of this sort of grounding before dealing with the formula $2\pi r^2 + 2\pi rh = 2\pi r(r+h)$.

Another simple use of letters which some pupils do not meet as early as they might is the generalisation $2n$ for an even number and $2n+1$ for an odd number, and similarly $3n$ for a multiple of three. It will be a new way of thinking about numbers for some, and practice can be given such as 'if $n$ is 5 what is $2n$? what is $2n+1$?', and also the reverse exercise of finding $n$ if the even number is, say, 38 or the odd number 53.

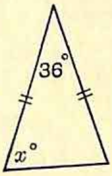
**6·315. Letters for unknown numbers.** This last exercise introduces the idea of using a letter to stand for a number which has to be found but for which an explicit formula is not given.

$2n+1=53$ is not arranged as a formula to find $n$ but as an *equation* which can be solved for $n$.

Pupils should make equations before they learn to solve them, but they will not at first find it easy to make statements about numbers which they do not know. 'Let's call it $x$ until we have found it' may be a somewhat informal but effective way of beginning. In finding the angles of triangles, perhaps,

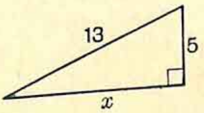


$x+28+114=180,$



$x+x+36=180,$

or when Pythagoras' truth about a right-angled triangle is known,



$x^2+25=169.$

These examples have been taken from geometry, but the answer to an arithmetic problem can also be called $x$: it will be a new idea to most pupils to write down a symbol for the answer as a beginning.

**6·316. Changing the subject of a formula.** In early examples both equations and formulae, then, use letters to stand for numbers which are eventually to be found. Except in 6·314 there has been practically no suggestion of actually working with letters or doing any algebraic manipulation. If the statement 'The length of the perimeter of a square is four times the length of one side' is expressed algebraically, with $p$ inches and $s$ inches as the lengths of the perimeter and a side,

we have $p=4s$

and we then deduce $s=\frac{p}{4}$.

Again we can rearrange $c=\pi d$ as $d=\frac{c}{\pi}$

or later $A=\pi r^2$ as $r=\sqrt{\frac{A}{\pi}}$.

Most modern school pupils will think of such examples as something quite fresh and unconnected with solving equations. Indeed for the above formulae about squares and circles it will be better to derive the second form direct from the figure, and not from the formula at all. But cases may occur where the subject of a formula has to be changed by general reasoning: for instance a boy who knows $v=\frac{d}{t}$ may want to find $t$ when $d$ and $v$ are known, and he may proceed in two steps, $tv=d$, and then $t=\frac{d}{v}$; or if a pupil knows that the average of $a$, $b$, and $c$ is $n$ he may write $a+b+c=3n$ and then deduce $c=3n-(a+b)$. Pupils should think these steps out for themselves and not be hurried into rules like 'Divide both sides by $v$'; 'Take the same thing from each side'.

**6·317. The formula as a relationship.** In time, too, pupils will come to understand a formula in a rather different way, not so much as a rule for finding something when something else is known, but as a relationship between two, or more, numbers derived from quantities each of which may vary, though the relationship persists.

Area of a square = square of its side.
Length of a spring $=a+bw$ where $w$ is the weight carried, and $a$ and $b$ are *constants.*

The convention by which some letters represent *variables* and others represent constants is always puzzling to beginners, even adult ones, and at first specific numerical constants should be used.

But the graphical study of a number of cases where $y$ varies directly as $x$, and of others where $y$ varies as $\frac{1}{x}$ or as $x^2$, as well as cases of the law

$y = ax + b$ should be well within the capacity of many modern school pupils (6·47).

At first however teachers may find it best to speak of 'the graph of $x^2$' or 'the graph of $\sin x$' rather than to introduce a second letter, $y$, to stand for the function. The graph of a function is a simpler idea than the graph of an equation, especially if pupils have been using equations to solve problems.

**6·318.** The fore-going paragraphs give in some detail the contexts in which modern school pupils may become familiar with the use of letters to denote numbers, but we would stress again that this familiarity will be gained in the course of all kinds of mathematical work, and not in a series of lessons specifically labelled 'Algebra'.

Gradually too pupils will come to realise that expressions like

$$a+b;\ p-9;\ \frac{k}{3};\ r+5;\ 5y;\ m^2;\ \frac{l+g}{2}$$

also denote numbers and that the plus and minus signs in them are not instructions to work a sum, or a difference, but a part of the 'answer'. This aspect of the use of signs is referred to in 6·153 There too it is recommended that at first $a \times 12$ should be used rather than $12a$, but when pupils come to use expressions which adults use e.g. $lb$; $2\pi r$; $\pi r^2 h$ they will need to understand that multiplication is intended, and they must also become familiar with $\frac{d}{t}$ instead of $d \div t$.

**6·319. Steps in manipulation.** So far we have considered only the writing and understanding of algebraic numbers and statements. If pupils are to take their knowledge any further they must discover, or rather be told, that some combinations of letter numbers can be altered in form without being altered in value. For the pupil who has struggled with spelling and found place value in decimal notation a difficulty it will seem puzzling that he may write either $lb$ or $bl$ for the area of a rectangle, that $12s+p$ or $p+12s$ can be the answer in a reduction to pence, but that $2\pi r$ and $\pi r^2$ are two very different things. Unless he has been allowed to proceed at his own pace, however slow it seems, and has been able really to understand and absorb each new idea, it is small wonder if he gets muddled.

Some steps in working he may take:

$$\begin{array}{lll} x+x=2x & \frac{y}{2}+\frac{y}{2}=y & \frac{a}{3}+\frac{b}{3}=\frac{a+b}{3} \\ 3a+5a=8a & 7b-3b=4b & \\ p-p=0 & \frac{c}{c}=1 & \end{array}$$

But manipulation should certainly begin at this extremely elementary level and be kept closely in touch with other work; although it is not difficult for pupils of moderate ability to obtain some early success in algebraic techniques under skilled instruction, these techniques tend to become meaningless habits, which are very liable to be forgotten as soon as practice in them ceases if they have no foundation other than memory of the teacher's instructions, and then mistakes are made in such things as ' collecting terms ', or ' cancelling an $a$ ' in an expression like $\frac{a+c}{a-2b}$.

### 6·32. Extension of the Idea of Number, including Directed Number.

Pupils who have come to use formulae will have little difficulty in accepting the fact that $a$ or $b$ or $r$ or $x$ may stand for a fraction. Thus if a length is called $a$ inches they will not be surprised to find that $a$ is 3·8 or $2\frac{3}{4}$ or $\frac{1}{4}$, and they will soon see that when $s=\frac{2}{3}$, $s$ shillings $=8$ pence. They will also be quite ready to write $\frac{p}{5}, \frac{3k}{7}, \frac{m}{q}$ and to speak of these as fractions, but such expressions as $\frac{x+y}{g+h}$ will prove more difficult, and should not be introduced unless and until they arise naturally, if they ever do.

If fractions occur in solving simple equations, say, $5x=29$ or $7y=3$, vulgar fractions should generally be used in preference to decimal fractions, since many simple vulgar fractions have no exact decimal equivalents; but there may occasionally be some over-riding advantage in giving a decimal answer, as for instance if several answers have to be added, or if the number is a trigonometrical ratio from which an angle has to be found. Recurring decimals are best avoided.

**6·321.** The use of plus and minus signs as the labels of directed numbers is referred to in 6·152 and 6·29, but the magnitudes there used are all expressed numerically and the signs are explicit. But $x$ may be used for a number which is a directed number, such as a number of feet above or below sea level, i.e. for both a magnitude and a sign. We do not however suggest that the modern school pupil should proceed to work with negative numbers even in addition and subtraction, except in particular cases which may occur in connection with other work, e.g. in dealing with logarithms. The teacher should note that the negative number occurring as the solution of such an equation as $x+5=2$ involves the general rule for the addition of directed numbers since $(-3)+(+5)=(+2)$. This equation will occur for the modern school pupil only when he is making a statement about quantities which are directed, for instance ' A rise of 5 degrees brings the temperature to 2 degrees above zero; the original temperature, $x$ degrees, was therefore $-3$ degrees for $x+5=+2$ '. On the other hand a statement which is incorrect or a problem which is false may be revealed by a negative answer; for example the false problem ' What is the vertical angle of an isosceles triangle each of whose base angles is 100 degrees?' gives the equation $x+200=180$ or $x=-20$ which is meaningless in this context, though positive and negative angles have meaning, counter-clockwise and clockwise, in problems of rotation.

That algebraic form supplies the urge to invent positive and negative numbers is too mature an idea mathematically to be within the scope of this report. Teachers who are interested will find it discussed in Appendix I of the Association's *Teaching of Algebra* where it is contrasted with Appendix 2 of which our own treatment of the generalisation of number is a much simplified version.

Some introduction to the relation of directed number to algebraic form may however be gained by extending such graphs as ' the graph of $ax+b$ ' to include negative values, and appealing to a geometrical continuity. Even in this the argument will be more effective if $x$ and $ax+b$ have some real significance for the pupil. Particular cases of $ax+b$ can of course be discussed, e.g. $3x+2$ or $2x-5$ or $7-x$.

### 6·33. Index Notation.

Pupils who can express 64 as $4^3$ or $2^6$, and recognise $10^5$ as a hundred thousand (6·15) may still have difficulty in realising that $x^4$ is only another way of writing $x \times x \times x \times x$ unless preliminary work in using the notation is simple and meaningful, and for this reason $x^2$ in area problems, and $x^3$ in connection with volumes are worth the most careful attention. It is well to accept $xx$ or $x \times x$ as a first stage, and to offer $x^2$ as a neater form rather than to say arbitrarily ' we do not write $xx$ but $x^2$ '.

**6·331.** What will prove more difficult is the realisation that it may be necessary to multiply $q$ by $q$ and to obtain $q^2$ when no question of area is involved; for instance $q$ yards of tape at $q$ pence a yard would cost $q^2$ pence. When a pupil meets such a number for the first time he may well feel that one of the advantages of using letters is disappearing; hitherto he has used one letter for a length and another for a price, and this has helped him to understand what he was doing. So not many pupils in the Modern School will be able to take this step, and in any case it should be deferred until it is required in some other work, e.g. $v^2$ in some formula for energy or resistance, or various powers of $r$ in the terms of a geometrical progression. Numbers of the type $k^4+3k$ are unlikely to be wanted at all.

**6·332.** The powers of 10 will however be wanted if, and when, logarithm tables are introduced (6·28); and some pupils may learn to make a graph of $10^x$ which will serve for finding logarithms, and, if extended to the left, will help to explain how a negative number can be an index.

### 6·34. Algebraic Form.

We have implied that pupils are most likely first to use letters to stand for numbers in giving a mnemonic form to a rule. The fact that letters are being used has been emphasised, but we have so far laid less stress on the form itself. Algebra is essentially written Mathematics: its vocabulary is small, but anyone who is to use it must be able to recognise and compare the various forms of expression.

Thus $A=lb \quad d=vt \quad k=pv$ are similar forms, each involves a product. The same three relationships can be written as divisions (in more than one way, of course), and assume the form associated with a rate

$$l=\frac{A}{b} \quad v=\frac{d}{t} \quad p=\frac{k}{v}$$

Other forms which may have to be recognised are ' difference of squares ' and $y=ax+b$. Symmetry of algebraic form is mentioned in 6·5313 in connection with symmetry in geometry.

### 6·35. Equations.

The equation is itself an algebraic form, and when the pupil first meets it he must learn to regard the sign = in a new way, and to write only one such sign in a line. This sign has often been inaccurately used in the Primary School, but it is not new to the pupils; what is new is that the expressions on either side of it are both still to be used in solving whatever problem is in hand, whereas previously he has used the sign only between one stage and the next of his working. A formula is, of course, an equation, but it is usually first met in a form ready for the insertion of numerical values on the right-hand side, and no manipulation has to be done. Thus a pupil will quite happily find a temperature in degrees Fahrenheit from the formula $\mathrm{F}=\frac{9\mathrm{C}}{5}+32$, but not from the same formula written as $\mathrm{C}=\frac{5}{9}(\mathrm{F}-32)$, and he will not readily derive one form from the other.

Pupils may first solve easy equations by inspection or trial, thus $2x+5=9$ is true if $2x$ is 4, or $x$ is 2, because if $x=2$, $2x+5=(2\times 2)+5=9$. Again $x^2=25$ has by inspection an answer, or *root*, $x=5$: the negative root is not in the pupil's experience at an early stage. In 6·316 it was pointed out that the general reasoning involved in the techniques of solving equations (or changing the subject of a formula) must be approached slowly.

Pupils who are quite accustomed to equations are often confused when one ' side ' is zero; they do not always realise that $x-7=0$ is equivalent to $x=7$. Solution of the equation by inspection or trial is more convincing than the form of general argument ' add 7 to both sides '.

### 6·36. Algebra Used in Other Branches of Mathematics.

Most of the examples of algebraic symbolism we have given will easily be accepted by the pupil as part of arithmetic or geometry, but Mathematics is a unity, and the same language of symbols is used in all its branches. Our recommendation that algebra should not be taught as a separate branch is in accordance with the practice of present-day language teachers in teaching grammar through the use of language, and not separately. But it does lay on the mathematics teacher the need to be careful in introducing new symbols and forms of expression to see that all his pupils understand each one, for usually the meaning cannot be deduced from the context as it can sometimes when a new word occurs in a spoken language.

**6·361.** In the section on trigonometry (6·6) there are warnings about the notation being new, but trigonometry cannot progress far without the use of formulae and equations. Even in the right-angled triangle we have $a=b\,.\tan A$, and the pupil who has not done any algebra may learn to use a dot as a multiplication sign. If he progresses to the formula $\sin^2 A+\cos^2 A=1$ there is a new use of the index to be learned, particularly puzzling if he has previously only used squared numbers in connection with area. Simultaneous equations demanding considerable experience in algebra will be involved if the problem of finding an inaccessible height by the method given in 6·63 is undertaken by abler pupils. As with many of the equations and formulae of trigonometry, the coefficients when expressed numerically are in decimals.

**6·362.** Pupils who become at all deeply interested in the topics of chapter 7, surveying, navigation, mechanics, statistics, require at least to be able to read formulae, some trigonometrical, some purely algebraic. If the language of algebra is familiar, pupils may gain some idea before they leave school of the part played by Mathematics in these subjects without being able to establish any but the simplest of the formulae which they find quoted in the books.

### 6·37. Algebra as a Vocational Study.

In 3·5 it is suggested that in the last year in school the Mathematics studied may be given a vocational bias. For pupils who hope to go on to a technical college, or to enter some employment in which a certificate in Mathematics is desirable, algebra may be undertaken as a vocational study. Teachers of such pupils should consult the Association's *Teaching of Algebra*, but we are confident that they will find that pupils who have understood the limited use they have already made of symbols in a course planned in accordance with the suggestions we have made will make rapid progress in the more formal work, when they undertake it with the knowledge that they will need it as a tool in their chosen life-work.

### 6·38. Additional Notes about Written Symbols.

It is advisable that pupils should use the written form of $x$, and not just a cross, but whatever form of letter is used it must be adhered to for the same number throughout a piece of work: if two forms of the same letter are used (as $R$ and $r$ in the formula $\pi R^2 - \pi r^2$ for the area between the circumferences of two circles, one within the other) they stand for different numbers. Sometimes letters are modified by the addition of a suffix, or in other ways, to make extra symbols: it is usual to call the radii of the escribed circles of a triangle $r_1$, $r_2$ and $r_3$ and the average of four numbers $y_1$ $y_2$ $y_3$ and $y_4$ is often called $\bar{y}$.

Brackets form a part of algebraic symbolism e.g. $a+b+c+d$ may be written $(a+b)+(c+d)$ to indicate that the expression is to be thought of in two parts instead of four. Different forms of brackets should be made if one bracketed expression forms part of another one, as for instance

$$a-[p\div(n+y)].$$

A tie, or line, over the expression is sometimes used instead of a bracket, thus, the above might be written

$$a-(p\div\overline{n+y})$$

with the same meaning.

The fraction line is sometimes extended to act rather like a tie when two fractions have the same denominator, thus $\frac{p}{r}+\frac{q}{r}$ is written $\frac{p+q}{r}$ and is of course equivalent to $(p+q)\div r$.

A note on the sign $=$ is included in both 6·16 and 6·35. Here we would stress again that it must be carefully used.

To check that $x=7$ satisfies $\frac{x+5}{3}=2x-10$ write $\frac{x+5}{3}=\frac{7+5}{3}=4$ and $2x-10=14-10=4$ but do not allow such a truism as $4=4$.

## 6·4. Graphs.

*6·40. A graph as a symbol.*
*6·41. The axes, dependent and independent variables.*
*6·42. Building up a graph from observations.*
*6·43. Distance-time graphs.*
*6·44. Space-space graphs.*
*6·45. Discontinuous graphs.*
*6·46. Histogram, frequency polygon, ogive.*
*6·47. Quantities connected by a mathematical relationship.*
*6·48. Using a graph.*
*6·49. Drawing a graph.*

### 6·40. A Graph as a Symbol.

Examples have already been given of the use of diagrams in which numbers or quantities not spatial in themselves are represented spatially by lengths, by areas or by the pictorial symbolisms sometimes called isotypes (6·217). These pictorial representations of number make a useful introduction to representation by length or area for the pictures suggest some of the properties of what is represented much as the old hand symbol for 5, a clenched fist, suggested the five fingers.

Mathematical ideas, which would be too difficult for modern school pupils if they were expressed solely through numbers and algebraic notations, can be approached by further development of this spatial symbolism. In particular the ideas connected with the related changes of two quantities can be approached through what we call a graph. Some details of the techniques involved in drawing graphs are given in 6·49; it is here assumed that the reader already knows their main characteristics.

The way in which the pupil becomes familiar with the graphical symbol follows much the same pattern as that for other symbols. Able pupils grasp its meaning and power fairly quickly; others need a patient approach stage by stage, not necessarily following any logical order but dealing with particular cases on their merits as they occur.

It is important that the pupil should have a complete grasp of the practical circumstances surrounding the early examples he studies. Several such examples are discussed below. Another helpful stage, and probably an early one with secondary pupils, is the noting of some graphical record in a book, newspaper or advertisement and the discussion of its meaning. Again some statistical records may be of interest, but too difficult to grasp in their listed or tabulated form. If the teacher were then to produce a ready-made graph of the data any ' trends ' of the variation would be more apparent and inferences would be made more easily. Experience of reading a graph is of great importance in becoming familiar with it as a symbol. So too, is the building up of a graph, stage by stage from actual records, as described in 6·42. The making of a graph from a set of listed statistics is less effective at the early stage because it can be a cumbersome process while the symbol is unfamiliar.

### 6·41. Axes and Variables.

Of the two related variables in a graph one *generally* appears *independent*, or predetermined in the sense that one selects convenient values of it. For instance, if some quantity varies from one time to another, its value appears to depend on the time at which it is noted and time is thus the independent quantity; the value of the *dependent variable* is then recorded at predetermined intervals of time, that is we 'plot its values against time'. Alternatively a quantity may vary from place to place and then place is the independent variable. It is conventional to represent the independent variable, particularly if it is time, along the line, or *axis*, drawn from left to right on the page. Usage has sanctioned a few exceptions to this convention, but deviations from it should be discouraged as it forms a part of the visual familiarity, its ' shape ', by which deductions from the graph are made easier.

### 6·42. Building up a Graph from Observations.

Two examples, both with time as the independent variable, will illustrate how familiarity with the quantities represented leads towards the use of a graph as a vehicle of ideas:

(*a*) The temperature chart.

An ordinary thermometer exhibits changes of temperature spatially by changes in the height of the column of mercury (or other liquid). Any pupils will be able to make, quite readily, pictures of their thermometer spaced out along a time line, and it is not a great step to isolate from the drawings the heights of the mercury column as the actual measured lengths or to some convenient simple scale. If more than one record is taken each day irregularities in the way temperature changes may be suggested. If the school is fortunate enough to possess, or to have access to, a continuous drum recorder, the idea of an irregular but *continuous change*, generally without any sudden alterations, will be strengthened. The record will also give material for discussion of daily ranges of temperature, referred to again in 6·45. Again, if readings of the mercury thermometer are taken twice daily, the continuous recorder will show how misleading it may be to join up the heads of the columns graphing these readings.

(*b*) The graph of the growth of a plant.

This graph contrasts usefully with the temperature chart since it pictures *continuous*, though probably irregular, *increase*. Time takes precedence of height as independent variable. A growing flower stalk gives quick and easily comprehended results although if the graph were being used as a tool for botanical investigation other more significant measures might be recorded. A very simple way of making the record is to move the plant along in front of the record paper noting the height each day. Pictures may be drawn, or these may be replaced by lines and then by points.

Unless prejudiced by previous experience of joining points by straight lines, it is probable that most pupils will suggest a smooth joining of the column heads, or points, as a picture of the plant's growth in the times intervening between the actual recordings. This smooth joining of the different heights is the more likely if the probable line is traced out by a finger before drawing any pencilled line. *Interpolation* of values between those actually recorded, as for example for the weekend, makes assumptions about rate of growth which are only tentatively shown by the curve drawn. Even so, the possibility of interpolation, which is a progression on the first idea of reading a graph, should be realised.

Comments by pupils on how the plant grows will probably show that in their minds there are notions of *rate of growth* based on an idea of the average rate found by comparing actual growth in equal periods. This is at once associated with the steepness of the graph between any two points. To analyse this idea and to give it precision is a mathematical aim to be pursued from such beginnings throughout a school course. No opportunity which a graph provides for taking another step in understanding rate of increase should be neglected. At the stage here described the impression gained will be mainly qualitative.

### 6·43. Distance-Time Graphs.

The growth curve just discussed is an example of the *distance-time graph*. The slowness of growth masks the fact that it is in fact a graph of a vertical motion which is being drawn. Many examples of motion records must be drawn before the pupil is fully familiar with the idea of a panorama in time and can keep it clear from confusion with a panorama in space. Examples from recorded statistics should precede any examination of the particular case in which the motion takes place at a given constant rate. For instance the distance-time graph might be drawn of a miler's performance on the track. Here the timing of standard distances is more likely to be given than the distances run in equal intervals of time, but this should not prevent time being represented on the $x$-axis thus giving the graph its conventional look in relating steepness and average speed. Information about 'running steadily' 'putting out a great effort' or 'a spurt 100 yards from home' may be used to guide drawing between recorded points. This could lead naturally to the consideration of the form of the graph when the speed of the runner (or the speed of a mechanically propelled vehicle) is unchanged.

Distance-time graphs in which stops occur and in which therefore there are sections parallel to the axis bring out both the time-panoramic quality of the diagram and the connection between slope and speed. The value of such a graph to the compilers of railway time-tables will become apparent by graphing the distance-time factors for express and goods trains on one diagram.

Again the journey illustrated in a distance-time graph may be shown by measuring the distances from a place on the route other than at its beginning: it will then be necessary to consider distances East and West (or some other pair of directions) of the selected place and so to introduce directed numbers. This would open the possibility of appreciating the graph of a to-and-fro motion. Similarly time before or after a given zero hour might be considered.

### 6·44. Space-space Graphs.

Among the great variety of graphs which may be drawn from statistics perhaps the space-space graph is of special interest since it links the graph and the scale drawing. For example the heights above sea level might be drawn for points along a line drawn on a map, or experimental work might yield some information about levels of a road or hill from which simple contours could be drawn. From these a picture of elevation along a section would be obtained, but the picture would be distorted unless the scales for height and for horizontal distance were the same.

### 6·45. Discontinuous Graphs.

In contrast to the continuous changes discussed in the preceding paragraphs examples of *discontinuous* records may be of value. Two examples are now given:

(*a*) Maximum and minimum temperatures.

The records of daily values shown on maximum and minimum thermometers may be conventionally shown by heights above a time line marked in days. The precise time of day when these temperatures occur is not known but if they are placed one above the other, there is shown a direct measure of the range of temperature for that day. No joining of points or column heads has meaning in this diagram.

(*b*) Times of sunrise and sunset.

Daily times of these may be plotted and the hours of daylight will then be shown in a similar way to the range of temperature in example (*a*). There is, however, one marked difference. The values plotted change regularly. If, for example, plottings were made for every third day, the two intervening values could be estimated by interpolation on the smooth curve joining the known points.

These two examples show the importance of understanding the practical background when drawing graphs of statistics if valid inferences are to be made.

### 6·46. The Histogram, Frequency Polygon and Ogive.

The frequency distribution is one example of the systematic arrangement of a collection of data. Its graphical representation may be familiar to children in view of its increasing use in illustrating economic facts in newspaper reports and in advertising.

Where the variable is not continuous, the information to be graphed consists of the numbers, or frequencies, of the cases found to have each possible value of the independent measurement. For example if the coins in a collection of dinner money were sorted out into heaps according to their dates, the number bearing each date would be graphed.

In the case of a continuous variable the number of cases falling in specified intervals is made. For example each child in the class could find the length of his span and the number of cases falling in the half-inch intervals, such as 6 inches and less than 6½ inches, could be determined.

The frequencies thus found may be represented graphically as follows:

(*a*) *By a Histogram*. This probably owes its origin to the bar or column chart as it consists of a number of adjacent bars of which the *area* represents the appropriate frequency. Since the bars are invariably drawn with the same width, their *heights* also give a measure of the different frequencies. The children can begin this by visualising the appearance of the piles of coins placed next to one another in order, in the example above. For the second example above, the idea of representation by a column will be apparent if the number of cases is shown by constructing a tally sheet, where the tallies are simply ticks or crosses, or the appropriate number of isotypes (6·40).

(*b*) *By a Frequency Polygon.* This is a variant on the histogram in which points are plotted with ordinates equal to the number of cases, the points then being joined by straight lines. Alternatively this polygon can be obtained from the histogram by joining the mid-points of the tops of the columns by straight lines and completing the polygon by joining the first and last mid-point to the beginning and end respectively of the horizontal axis. Pupils will be able to appreciate, just as in the cases of 6·42, that it might be possible to replace the straight lines joining the successive points by a smooth curve. This would be clear if, for example, the items were frequencies based on the number of marks in examinations obtained by a large number of children, or on the heights of a large number of children of the same age. The greater numbers would allow a smaller interval to be used on the base axis, and the suggestion of a smooth line to replace the polygon would be strong. Further, if the distribution were fairly regular, the frequency curve would have a shape reminiscent of that of a bell, and there would be a decided *mode* or ' most fashionable ' value.

In fact there would be enough evidence to justify mentioning the *curve of normal distribution*, showing its bell-shape, discussing the meaning of its mode and the value at which it appears, and stressing not only the scientific value of the curve, but its value in the commercial world as well.

(*c*) *By an Ogive or Cumulative Frequency Diagram.* This is a convenient way of expressing frequency distribution in a form which shows the total number of cases which lie below various values of the measurements. This is obtained easily by adding successively the separate frequencies determined for each measurement, that is, obtaining the cumulative frequency up to and including the particular value of the variable. The graph of the cumulative frequency plotted against the variable determining it, is called the ogive. Such a graph based on the data from examination marks would show at once the number (or percentage) of children below (and therefore also the number above if that were the information required) specific marks.

### 6·47. Quantities Connected by a Mathematical Relationship.

Where changes in two quantities are connected by a mathematical relationship we have the meeting place of the graph and the algebraic statement, as for example, in the graph of an equation with two variables. The equation may not at first be explicit but experience with the graph should lead, at the least, to the possibility of a generalised form of the relationship being discovered. For example points showing the calculated areas of squares for sides of different lengths may be plotted; this gives of course the graph of squared numbers which may or may not be summarised as $y=x^2$. Plotting the particular case of an area problem makes it easier for the pupil to accept different scales for $y$ and for $x$, if this seems desirable. The graph for the area of a circle, found by square counting, as the radius increases, will also appear to be a smooth curve concave upwards through the origin, when the curve of best fit is drawn through the experimental values. If drawn on the same scale as the first curve it will however be steeper. Discussion may suggest a multiplier. This can be tested numerically by dividing the areas by the squares of the corresponding radii, when a number differing slightly from $\pi$ will result in each case. Bad experimental values will be shown up by greater differences from $\pi$. The graphical check of plotting the circle area against (radius)$^2$ would give a straight line whose slope $\left(\frac{y}{x}\right)$ is $\pi$ i.e. the line $y=\pi x$ where $x=(\text{radius})^2$.

Some pupils may carry on beyond this experimental stage but the various lines of development are outside the scope of this report. It may be noted, however, that concentration on direct proportion, that is, on the straight line $y=kx$, may defeat its own ends. The importance and power of this graph (6·48) is made more vivid by contrast with that of $y=kx^2$, above, and with that which shows inverse proportion, that is $y=\frac{k}{x}$.

This latter type is constructed quite easily by plotting, for example, the length against the breadth of all the rectangles which can be constructed using 48 one-inch squares of card, or one factor against the other in every pair that can be found for 72, or the speed against the time taken for a journey of 120 miles.

**6·471.** The abler pupil may take some first steps in drawing graphs of relationships stated algebraically; the full value of this work cannot however be experienced until the $y$ and $x$ are directed numbers. Thus the curve $y=x^2$ considered above consists of one half only of the parabola through the origin; the other half is produced when $x$ is negative.

**6·472.** The real function of the graph in the Modern School is to provide a language in which to describe related changes between quantities met with in experience. This throws responsibility on the teacher for providing subjects worthy of exploration. Other subjects of the curriculum may often provide these. The kinds of investigation described in chapter 3 will give fruitful occasion for diagrams and graphs. The practical background will in the first place help the understanding of the graphical symbol. Finally the graph will come to be a powerful means of description and analysis.

### 6·48. Using a Graph.

A graph is drawn for a purpose, not for the sake of drawing it. Some of these purposes are summarised below:

(*a*) *To display information.* Statistics or algebraic statements are difficult to grasp as a whole; a graph presents relationships in an effective immediate form.

(*b*) *As a source of information.* Statistics are used to draw the graph and they are, as it were, stored in it and may be recovered by reading back pairs of values from it.

Additional information may also be obtained in favourable circumstances. The continuity of the graph may make it allowable to *interpolate* values not given in the original information, that is to estimate the values for points on the graph lying between known values. It may be possible in some cases to *extrapolate*, or to find values outside given ones, by continuing the curve beyond its original limits by considering the trend of its shape.

The *rate of change* of one quantity with another may be estimated with increasing precision as understanding of its relation to the slope of the graph is deepened from its qualitative and intuitive beginnings.

(c) *As an aid to calculation.* Generally graphs used to aid calculation are those in which two variables are related in a way that can be expressed by an algebraic equation, although they may have been drawn by using information that was given in some other form. For example graphs which are meant to be ready-reckoners for cost and quantity are straight lines which are drawn by calculating two pairs of values and joining the points plotted from these by a straight line. Here, although the equation of the type $y=kx$ is not given, it is assumed that this relationship holds good, i.e. that there is a constant cost per article (or per measure). A straight line graph is often used for converting one unit to another, as for example to convert kilometres to miles, or degrees Fahrenheit to degrees Centigrade.

Because it is easy to construct, and also because of the certainty of reasonable accuracy, a straight line graph is most often used in this way. If, however, enough points are plotted to make a free-hand drawing satisfactory, curves can serve the same purpose, that is as an aid to calculation. For example if against the numbers 1, 2, 3, 4, etc. their squares 1, 4, 9, 16 are plotted, and a few other intermediate points taken such as 1·5, 2·5, 3·5 with the corresponding squares 2·25, 6·25, 12·25, it is possible to draw a reasonably smooth and accurate curve from which a fair approximation can be found to the values of, for example, $2{\cdot}9^2$ and $\sqrt{12}$.

### 6·49. Drawing a Graph.

(*a*) A graph is a record which must have a *title* explaining what the changing quantities are, as for example, ‘ The Growth of a Plant from (date) to (date) ’, ‘ The Heights above Sea-level along a Section A to B ’, or simply, ‘ $Y=X^2$ ’.

(*b*) A graph must have two *axes* which must be labelled to show which quantity is measured along which axis.

(*c*) Unless a graph is a sketch of the relationship only, it must have two *scales*, one for each quantity (i.e. for each axis). If desired, these may be shown by marking the axes as well as, or instead of, stating the scale. The use of the expression ‘ small squares ’ as in ‘ 5 small squares to 1 mile ’ is not strictly correct; it is better to say, remembering that scale is concerned with length, ‘ 0·5 inches ’ or on tenth-inch graph paper, ‘ 5 small divisions ’.

The choice of scale is the most difficult part of graph drawing for the beginner. If the graph is one of statistics and the numbers are large, there is likely to be confusion unless some preliminary work has been done on the lines suggested in 6·216 or 6·217. The habit of noting the range of numbers required on each axis should be acquired. The omission of an unused part of the numbers, as in the height diagram in 2·131, that is omitting to take one axis as far as zero, is a useful device to make a larger scale possible.

(*d*) The teacher should not leave the actual *draughtsmanship of graph drawing* to chance but should consider with the pupils how the graph may best be drawn. This depends to some extent on the nature of the graph both in respect of plotting points and of joining them. For instance experimental values may be treated differently from calculated ones. From the former the children will be able to see the reasonableness of using ' a line of best fit ' as the resulting curve.

## 6·5. Geometry.

*6·50. Three aspects of geometry; the practical foundations of the course; some main objectives; means of learning.*
*6·51. Lines and angles.*
*6·52. Length, area and volume.*
*6·53. Shapes, two and three dimensional; symmetry; plane shapes bounded by straight lines; prisms and cylinders; pyramids and cones; the circle.*
*6·54. Similarity and congruence.*
*6·55. Loci and envelopes.*
*6·56. Conic sections.*
*6·57. General ideas in three-dimensional geometry.*
*6·58. Generalisation in geometry; the argument from measurement, from symmetry, from orderliness, from the necessities of construction; generalisations not readily discovered by the pupil.*
*6·59. Some general considerations; riders.*

### 6·50. Three Aspects of Geometry.

For the ordinary person, geometry is the study of spatial properties. Many pupils who have little appreciation of numerical properties and processes discover an unexpected interest and pleasure in investigating the world of space in which we live and in which objects of diverse size and shape occupy a definite position, or move in an observable path.

The importance of a study of geometry can hardly be overestimated today when travel, machinery, building construction, furnishings, dress design, horticulture, and indeed the structure of the Earth and its place in the Universe all call upon us constantly to use geometrical ideas. Considered in these terms geometry makes a powerful appeal to adolescent boys and girls. But both constructional skills and a mental picture of the space-world are dependent on the growth of spatial perceptions. Such perceptions can only grow through ample spatial experiences.

At all stages of the Modern School one main consideration should be borne in mind: **The geometry to be taught must be based on practical spatial experience.**

Three aspects of geometry can be distinguished:

(1) Perception of properties of shape, proportion, relative position, etc. which develops from observation and experiment, and leads to a recognition of principles and an appreciation of orderliness and beauty of form.

(2) Ways of measuring and calculating length, angle, area, etc. which are based on spatial properties. These help to deepen understanding as well as serving practical purposes.

(3) Opportunities of reasoning and making a series of deductions from the facts of observation and experience. These may be simple linkages of one fact with another or the perception of a relationship which enables a length or an angle to be calculated from known measurements, but in their essence they are steps of logic.

**6·501. Foundations in experience.** There is detailed discussion in the Association's *Mathematics in Primary Schools* of the ways in which a child begins to develop powers of talking and thinking and writing about space relationships and geometrical facts and truths. It cannot, however, be assumed that the entrants to the Modern School have gone far in this development or indeed that any secondary school entrant has had adequate basic experience on which to found a systematic study of geometry. Other reports of the Association (First and Second Reports on *Teaching of Geometry in Schools*, 1923 and 1938, *Teaching of Mathematics in Preparatory Schools*, 1924) all distinguish a first stage in which the work will be experimental and associated with arithmetic and other studies such as geography. Land measurement is noted as a likely source of geometrical experience. (This is elaborated in *Mathematics in Preparatory Schools*). Where crafts play any large part in a school these would of course also give many opportunities for geometrical experience.

Valuable discussion and examples of this experimental stage, called Stage A, are to be found in these earlier reports. In many text books however the work of the stage tends to be planned solely as an introduction to systematic deductive geometry. This treatment is unsuitable for the modern school pupil who may only make brief excursions into the field of deduction and must rely on more practical experiences throughout his study of geometry. It is therefore proposed in this section to make a fresh approach to the work of Stage A and the possible developments from it, and to recommend the interested reader to supplement what is said by reference to the four earlier reports.

**6·502.** For the modern school pupil the value of experimental work in geometry will depend to no small extent upon the choice of practical themes. The risk of work in geometry becoming desultory or time-wasting is greatly reduced if background experiences are chosen which are in themselves worth while and interesting. Examples are to be found in 7·1, 7·2 and 7·3 in which the background study in itself forms a course of work; such purposeful studies are particularly helpful in avoiding the desultory and in revealing a connected body of geometrical truths which are relevant to men's work and conditions of life.

The geometry included in the course should be related to matters with which the pupils have some acquaintance. The appreciation of geometrical truths and the effective use of drawing and measurement should grow out of the situations and problems in which the pupils have, or can develop, an interest, and not be arbitrarily imposed as an academic discipline devoid of other value. For example, the value of accurate measurement and drawing may be realised through its effect in more careful dressmaking, in the pleasing lay-out of a garden, or in the exact setting-out of an athletics course. But the study should not be confined to utilitarian topics or those which appeal only to boys and girls who delight in craft; all pupils should have opportunities to experience the aesthetic and intellectual values of geometry to the full extent that their individual abilities allow.

**6·503. Coordinating different aspects of geometry.** Although we have distinguished three aspects of geometry (6·50) it is not intended to suggest that there should be any such division in the mind of the pupil nor in the teacher's planning of a syllabus. Any geometrical study may involve all three. In the planning of a garden choice of shape and judgment of size arise and there is opportunity for making short deductive steps; all are in addition to the obvious need of calculations in such an enterprise. The division into three aspects was made in order to call attention to the variety of possible approaches to the discovery of a geometrical truth. The teacher who is aware of all these aspects will be able to enrich his teaching by a greater width and liveliness. For the pupil it is important that he should increasingly see geometry as a related and consistent whole and as a coordinated part of his total study of Mathematics.

A purely geometrical theme may be handled in the same three fold way. The discussion which leads to the discovery of the rule for finding the area of a triangle springs from experiment and develops by argument into a generalisation about all triangles. The rule is then used to find the area of particular triangles; formulation follows and finally the formula is used to find triangular areas of many kinds.

**6·504. Some main objectives.** It will be seen that we recommend that the syllabus for the Modern School shall not be developed in a strictly logical order, each step being dependent on what has already been learnt, as is possible for the ablest pupils in other schools. Instead, the teacher must think out the geometrical content of the four-year period in new terms and with clearly defined aims. We suggest that he should try to give his pupils opportunities to acquire:

(*a*) an appreciation of shape, both in plane figures and solid bodies, such an appreciation to include arrangement of the shapes relative to one another. This will necessitate the growth of the ability to compare by ratio rather than by difference, and should help in understanding the numerical significance of ratio;

(*b*) an appreciation of such properties as symmetry, smoothness of curvature, parallelism, and of such relationships as congruence, similarity;

(*c*) an ability to produce a desired shape working from a plan, a model, an imagined object, or the development of a solid; conversely, to devise from a solid a plan which would enable a copy to be made;

(*d*) an understanding of geometrical terms commonly used;

(*e*) some skill with tools such as the ruler, compasses, protractor, clinometer;

(*f*) some knowledge of methods of comparing and determining size in length, area, volume, angle;

(*g*) some power to envisage a locus, an envelope, or a solid generated by a moving area;

(*h*) an ability to recognise a triangle or other figure in space where its sides are not visible. This power to visualise a configuration has many uses, not only in practical work such as surveying, but also in the building up in the mind of a spatial structure which enables the pupil to think about his world geometrically.

**6·505. The means of learning.** The forms of expression to which reference is made in 6·1, i.e., spoken and written words, pictorial and diagrammatic drawings, modelling in various materials, movement and symbolism, will all be used in the study of geometry, and the reader is advised to re-read 6·1 in connection with this section because of its particular importance in geometry.

**6·506.** There are in common use a considerable number of geometrical terms to many of which 6·111 refers. Whilst the understanding of these terms should be part of the equipment of the ordinary citizen we strongly deprecate the deliberate teaching, in an academic way, of a vocabulary, or of verbal expressions, apart from their use in relation to a pupil's own experience. On the other hand, mathematical words which are required to describe such experience should be used accurately. Certain topics discussed below will by their nature call for many new words. When a pupil reaches awareness of a new concept he feels a need for new words in which to talk about it. Accuracy in use depends on the extent to which the concept has developed. Some pupils will have acquired at the primary stage a limited geometrical vocabulary but will require further experience in order to expand it and to refine the meanings. All pupils will have vague ideas of the meanings of such words as *level*, *vertical*, *sphere*, *dimensions*, but after experience and discussion will come to attach to them more strictly mathematical meanings.

It is very important that pupils should have opportunities of making statements about their spatial discoveries in their own words; they should not be given a form of words by an adult until they have become dissatisfied with their own attempts and are ready to accept the new words as an improvement. Pupils should frame their own definitions.

It is likely that the teacher will introduce a new idea with the use of phrases which are not yet within the pupils' written or spoken vocabulary, making the meaning clear by demonstration or discussion. In such cases it is important that the teacher uses the words with mathematical accuracy in order that the level of the pupils' own speech be thereby raised. Any lack of precision in the teacher's use of the words is likely to be imitated by the pupils.

**6·507. Some examples of drawing and modelling.** By its very nature geometry will involve many diagrams and drawings. These are of special value to pupils who find the verbal expression of abstract ideas difficult yet can readily interpret a pictorial representation. So important is drawing that 7·3 is entirely devoted to it. We therefore turn now to the varied practical activities in which pupils take pleasure and through which concepts are developed and properties discovered. The list below is illustrative only; teachers will find many other activities which fit the interests of their particular classes.

Spacing shapes, for example mounting a photograph, spacing buttons on a dress, plants in a garden bed, or furniture in a room.
Arrangement of shapes such as bricks in a wall, tiles on a floor. (See also 6·53.)
Drawing patterns, freehand, on squared paper or using instruments (6·53).
Making a model in connection with history or geography, in which shape rather than scale is important (6·574 etc.).
Discovering the ' pattern ' which can be used to cut out a desired object, for example a lampshade, a circular skirt.
Making scale plans and models, for example, of classroom, school, playing fields.
Enlargement and reduction of maps and drawings (6·54, 7·335).
Practical ways of constructing right angles (6·512) to use for setting out the points of the compass, laying out a games pitch, testing an angle.
Making a protractor, a clock-face, a mariner's compass, a clinometer, a sextant.
Studying and using the orientation marks and scales of maps (7·2).
Making or using models to show the meaning of latitude and longitude.
Making a moving model such as a winch to show the effect of rotation.

**6·508.** In addition to the activities set out in the previous paragraph, the broad topics of chapters 3 and 7 afford valuable means of discerning and applying the truths of geometry. In the rest of this section we consider the study of geometry from two standpoints: the facts and concepts which should form the structure of the course, together with additional topics not often included, and also the ways in which pupils become assured of the truth of such ideas and can generalise them into principles.

### 6·51. Lines and Angles.

A pupil readily apprehends a line as a path traced out by his pencil on paper, and he recognises that it may be either *straight* or *curved*, but his use of these words may be somewhat vague; in common usage the word *straight* is sometimes associated with direction, as in the phrases *straight up*, *straight on*, and the geometrical meaning of this word will therefore need to be made clear. This meaning will include several ideas:

(*a*) that a single straight line may be ' drawn ' through any two distinct points
(*b*) that it may be continued beyond them in either direction
(*c*) that innumerable other points lie on it
(*d*) that a third point chosen before the line is drawn may be on it or may not.
(*e*) that a *line of sight* can be imagined so that the idea of a straight line can be isolated from a drawing 7·11.
(*f*) that plane surfaces meet in a straight line.

The idea that one straight line may go on and on without beginning and without end is an extension in thought from the straight lines which are already familiar as the sides of squares and rectangles and triangles, but which have definite end-points and definite lengths. Later, the pupils will draw wave curves (sine curves) and others which have this unending property. They will compare these with the completeness of a circle or ellipse which encloses an area; a pencil used to trace out a circle returns in course of time and passes again and again over the same path, but in drawing a straight line or a sine curve each point and each piece of path is passed over only once.

**6·511. Angles.** Some teachers introduce pupils to the angle through the idea of a corner, ' sharp ' or ' right ', but it is doubtful whether, in fact, the corner is anything more to the pupils than a sector of a plane. Observations of the changing shapes of a jointed framework, e.g. four Meccano strips joined at their ends, may lead on to the more dynamic idea of an angle. It will be seen as the result of a rotation of one line moving away from a first fixed line, as when one leg of the compasses is held and the other opened. Later, two lines rotating simultaneously, the angle between them changing, will be observed, e.g. the hands of a clock. When diagrams are made of, for example, a clock-face and the hands, the angle is seen as formed by two intersecting lines in a static way.

The *right angle* is seen to have special properties: four make up a complete turn, two adjacent right angles make a straight line, or ' straight angle '; the corner angles of most objects in the classroom are right and, in common usage, the right angle is understood in the order ' right (or left) turn '. In fact, movements of arm and body are readily used to show ' turning through a right angle '. Association of the word ' right ' with ' upright ' and ' forthright ' will avoid the mistake of talking about a ' left angle ' for the contrary turning.

Through discussion and observation pupils will become aware of angles of different sizes and will, in comparisons with the right angle, learn to distinguish angles which are *acute*, *obtuse* and *reflex*. Consideration of the rotation of a moving arm, e.g. the minute hand of a clock, or the spoke of a revolving wheel, may give rise to the question of the size of angles greater than a complete turn, and the meaning of *revolutions per second* can be explained.

**6·512.** It is advisable not to hurry the early part of the work as the concept of an angle is not an easy one to assimilate. Useful and interesting work may be done with right angles and their halves and quarters, before introducing the degree and the protractor. Some applications are given:

Paper-folding to get the right angle, its half and quarter, thus giving each pupil an individual measure for use.

Construction of right angle with compasses or with set square for the accurate drawing of patterns and plans.

Meaning of horizontal, vertical, parallel; distinction between vertical and perpendicular (6·112 *b* and *c*). Pupils will be interested to see the plumb-bob and spirit level. Rectangular plans and elevations, and marking the position of an object upon them, use these ideas.

Making a 3, 4, 5 triangle from knotted string, thus giving a right angle useful for the lay-out of a larger rectangle out of doors.

Construction of a mariner's compass card showing 8 points and the use of this to find directions (and, with pacing or tape, the lengths of routes on ground or map). This is an extension of primary school work in which the compass point is probably only the *name* of a direction.

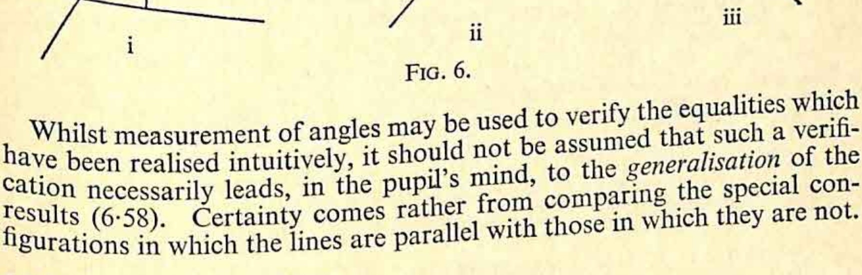
Construction of a simple sectograph, i.e. a circle divided into sectors to show parts of a whole (6·241).

With the introduction of the degree a wide field of application is opened up. In the suggestions of 6·507 and the topics in chapters 3 and 7 many opportunities are offered for the fostering of an understanding of angles and their practical uses.

**6·513.** Many of the generalisations about *equal* angles are intuitively understood and accepted, and may be approached almost as a matter of language training. Thus when two lines cross, the *vertically opposite* angles will readily be accepted as equal, and any pair of adjacent angles as *supplementary*.

Again, when a line cuts two other lines, two *corresponding* angles (or F-angles) such as *a*, *b* can be used to compare the directions of those two lines (Fig. 6 i), and the fact that *parallelism* and *equality of corresponding angles* go together should easily be realised.

A similar statement using *alternate* angles (or Z-angles) such as *c* and *d*, or *e* and *f*, follows in the same way, though it may be a little less obvious (Fig. 6 ii and iii).



i ii iii

FIG. 6.

Whilst measurement of angles may be used to verify the equalities which have been realised intuitively, it should not be assumed that such a verification necessarily leads, in the pupil's mind, to the *generalisation* of the results (6·58). Certainty comes rather from comparing the special configurations in which the lines are parallel with those in which they are not.

### 6·52. Length, Area and Volume.

A line has length, a surface has area, and the three-dimensional space in which we live has volume. Definitions of these terms in any precise or abstract form are out of place in the Modern School, and indeed with any young pupils, but boys and girls will use the terms with accuracy as they gain experience of comparing length with length, area with area, volume with volume.

**6·521. Length.** For an understanding of length the pupil will need to have accepted the following ideas:

(*a*) that a *length of line* stretches from one point to another,
(*b*) that two lengths of line may be equal,
(*c*) that there can be a length of curved line but that this is longer than the straight line joining its end points,
(*d*) that a path may consist of a series of straight lines, as, for instance, the perimeter of a hexagon.

The concept of length, or segment, of line definable by end points grows with the appreciation of decimal notation. For example, a length of 3·7 inches may be approximately measured as 3·65 inches; 3·69 is nearer to 3·7; 3·695 and 3·699 are progressively nearer still. This sort of consideration helps to crystallise the idea that there are end points to this line segment which we can designate 0 at one end and 3·7 at the other, on the unending line of which the segment forms a part.

No formal teaching, or even drawing together, of such ideas is required but the teacher must be aware of what is involved, and ready to seize opportunities for nurturing his pupils' perceptions.

**6·522. Area.** Though most pupils will have heard of area and probably had some lessons on it in the Primary School, ordinary children beginning the secondary stage of education will require to study it as a measure of surface in its fundamental aspects, without first using conventional units. Several possible opportunities for its introduction are mentioned in this report, e.g. through gardening in 3·41 and housecraft in 3·42. In 4·811 a remedial approach is outlined which is suitable for many first-year pupils.

In this section we set out, though not necessarily in a teaching order, some of the geometrical experiences and ideas which underlie the understanding of area.

(*a*) Recognition that two amounts of surface can be compared by superposition; if they just fit they are equal; if one is entirely enclosed it is smaller; if it entirely overlaps it is larger. The cases of partial overlap will give rise to uncertainty (Fig. 7). Suggestions may be made for division into smaller pieces. For this work templates are useful for outlining portions of surface on paper. They can then be used in variety for superposition. Pupils will probably observe in passing that pieces with the same shape may have different areas.

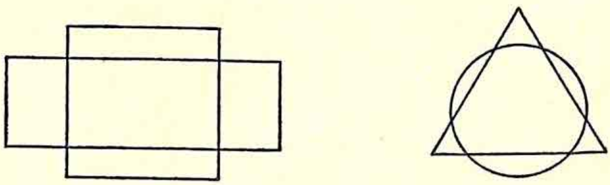


FIG. 7.

(*b*) Areas can be added or sub-divided. In Fig. 8 the shaded sectors can be added and compared with one another and the whole; areas 2/8, 3/8, 3/8 of the total, can be identified by superposing

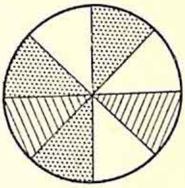


FIG. 8.

a movable sector. Similarly a movable piece can be used to test the equality of the shaded and the unshaded portions in Fig. 9. It will be noticed that equal areas may have different shapes. This can be brought out, as in Fig. 10, by covering them with the same number of identical shapes.

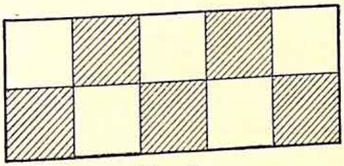


FIG. 9.

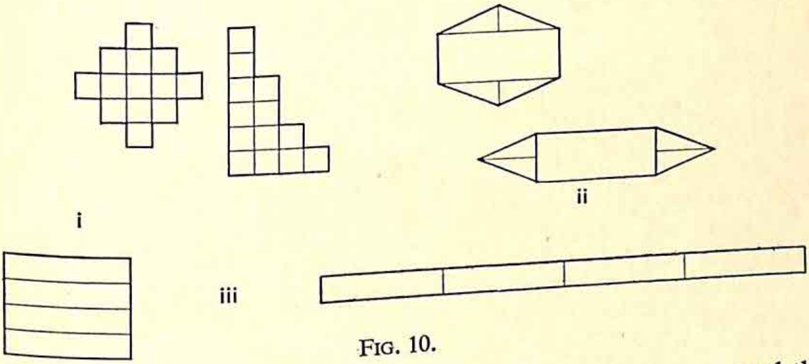


FIG. 10.

A number of equal squares can be cut up in various ways and the pieces re-arranged to form different shapes (tangrams), as in Fig. 11.

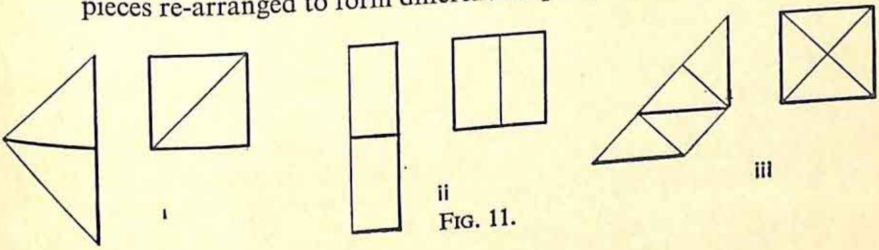


FIG. 11.

(*c*) Comparison of areas of familiar things such as table tops or shelves by counting the number of round tins or rectangular packets that can be packed closely upon them. It will be observed that round shapes leave gaps uncovered; rectangles leave no gaps.

(*d*) Comparison of two irregularly shaped pieces, e.g. of land; the questions 'How many cabbages would they grow?' suggests counting patches, either round or square; the square patches would leave no gaps between them.

(*e*) Construction of a square inch and a square foot, and their use in covering rectangles, ignoring at first any excess over a whole number of units.

(*f*) Finding the number of squares which would cover a rectangle by noticing the number in a row and the number of rows. This notion is quickly grasped intuitively but should not be understressed since it is the essence of the argument. After some experience with this semi-practical method the rule can be formulated and practised. Fractional sides can be easily dealt with geometrically in two stages, first with one side a whole number, then with both sides fractional. See Fig. 12, and also 6·246 on the multiplication of fractions.

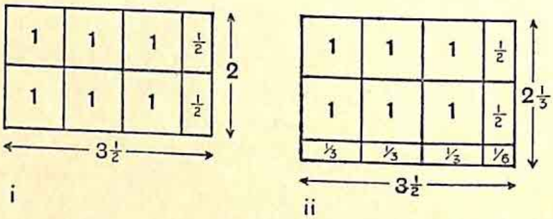


FIG. 12.

(*g*) Relating units to one another by fitting square feet into a square yard. The examination of a series of squares, fitting them into the corner of a larger square, and noticing the square numbers, 4, 9, 16, etc., will help pupils to visualise 144 square inches in a square foot.

(*h*) Work on squared paper to see $\frac{1}{10}$ and $\frac{1}{100}$ of a square inch; this should precede finding areas of rectangles with sides measured in decimals or those giving results in sq. chains, sq. links, or sq. cm.

(*i*) Use of rectangle formula for various shapes which can be rearranged as sets of rectangles or squares. Applications to the faces of prisms, cubes, etc.

(*j*) Experiments to establish methods of finding the areas of the triangle, trapezium, parallelogram, circle. Some pupils could learn the method of reducing any rectilinear figure to a triangle of the same area.

(*k*) Estimation of irregular areas by counting squares in a drawing on squared paper. This could be taken at an early stage if desired.

(*l*) Consideration of the area of curved surfaces. Realisation that a piece of paper can be stuck on the surface of a cylinder or cone to show that the curved area is equal to a flat one; the surface of a sphere cannot be shown equal to a flat one, but it can be painted and the formula for its area quoted.

**6·523. Volume.** The first interest in volume, the measure of 3-dimensional space, is shown in relation to capacity, the amount that a hollow vessel can hold, say a watering-can or a teacup. This is traditionally measured by comparison with the capacity of another vessel holding, say, a gallon or a pint, and is found by pouring the contents, possibly corn or milk, from one to the other. This process is independent of shape and does not allow of calculation in terms of length. When it is required to find the amount of space occupied by a *solid* body some means of measuring other than pouring from one vessel to another must be found. If the new method could be based on length measures it would have the advantage of making it possible to calculate theoretically the volume of a solid known only in terms of its surface measurements.

Confusion may arise about what is meant by the phrase 'volume of a solid' in the case of a *hollow* body, e.g. a wooden box; a clear distinction in language should be made between the internal capacity of the box, or *cubical contents* if the capacity is measured in *cubic* units, and the volume of the wood which forms the box.

The usefulness of a cubic measure can be seen by pupils interested in the volume of water contained in a full rectangular aquarium tank. If this volume could be stated as the number of bricks of a certain size which would build up a block with the same measurements it would not be necessary to empty the tank in order to find its internal volume. If the tank is already empty its volume can be found in 'brick units' by filling it with bricks; a comparison can then be made with the number of pints required to fill it. The connection between the two kinds of volume measurement can then be clearly seen.

Valuable experience of the 'brick' method for *solid* bodies is gained by arranging, say, 96 bricks in a number of different stacks, which will of course have equal volumes, e.g. 3 layers of 32 bricks, each layer having 4 rows of 8, or 4 layers of 24 bricks, each layer having 4 rows of 6. A block of 96 bricks can also be looked at as consisting of vertical slices; thus 3 horizontal layers of 32 bricks may be regarded as 8 vertical slices of 12 bricks. This is the geometrical equivalent of the algebraic statement $(a \times b) \times c = a \times (b \times c) = b \times (a \times c)$.

For solid bodies the connection between cubic and capacity measures is most readily seen if a block of, say, paraffin wax, is melted and poured into a graduated measure. Conversely a pint of fluid jelly can be poured into a rectangular mould to form a measurable block.

Through such experiments pupils will observe a number of facts:

(*a*) the volume is unaltered if the stack of bricks is divided into two or more stacks, or arranged on an L-shaped base, etc.

(*b*) a cube is the most convenient shape of brick for measuring volume because it enables a count of row, layer, and number of layers to be made in terms of the units in the length, breadth and height of a rectangular block, or tank.

(*c*) if the number of layers is doubled, trebled, etc. the volume is doubled, trebled, etc.

(*d*) the graduation of a cylinder, e.g. a jam-jar, in cubic inches or cubic centimetres can be demonstrated by pouring from a hollow cube. The connection between pints and cubic centimetres or cubic inches can be shown in the same way (3·512).

(*e*) the volume of a solid can be found by reading off the volumes indicated in a graduated vessel before and after the solid is immersed in the water it contains. This displacement method can be used both for finding the volumes of irregular solids and for experimental checks on computed volumes.

The power to visualise an empty space, such as a room, as built up of invisible bricks is encouraged by the practical activities suggested above. A wire framework which outlines a cubic foot is a further aid to imagination; it can also be used to show how large a unit a cubic foot is and how much can be stacked into it.

This detailed statement of preliminary work on volume is given because we are of opinion that the importance of such an experimental approach is insufficiently realised. Computation of the volume of a rectangular solid of given or measured dimensions is less valuable than understanding the above and related ideas. When pupils are ready to use ' rules ' they will make their own formulae for the volumes of such regular or easily recognised solids as they require.

### 6·53. Shapes, Two and Three Dimensional.

Pupils will already have a large experience of shape which they have been acquiring since infancy. This experience will be extended, analysed and classified by discussion, drawing and modelling and the relevant vocabulary learnt as required.

Collections may be made by the pupils of examples of the shapes they have learnt to recognise. Some fruitful sources of the descriptive type of geometry are walls, gates, trellis-work, railings, pavements, garden plans, decorative details on doors and furniture, Gothic and other styles of architecture, leather work, patchwork quilts, plaster work, the many shaped tins and boxes of the shops, houses, chimneys, churches, woven materials and some printed materials. Pupils will enjoy finding incongruous members of ' families ' such as drain pipes and pencils. They will be interested too to note that the rectangle prevails in man-made shapes but not in natural ones; there is a link here with biology. The prevalence of circular shapes in the kitchen leads to a consideration of matters which are concerned with housecraft and craft.

Useful practice in the handling of tools and familiarity with plane figures may be gained by the construction of patterns. The discussion of these patterns, of the design of a length of dress or curtain material, suitably chosen, or of the shape of a tea-pot may lead to ideas of symmetry and proportion.

By his efforts to make a model of a church or to draw some architectural outline or detail, such as a Gothic style window, a pupil may become more alive to the geometrical aspects of his environment and so develop an appreciation of it.

**6·531. Symmetry.** Symmetry appears in many forms in almost every aspect of life. Its aesthetic value is a subject in itself; why we may like a symmetrical façade but not a symmetrical tree or mountain is not our concern here. Nevertheless we miss much if we fail to appreciate the beauty of mathematical symmetry, both of geometrical and of algebraic forms.

From the mathematical point of view we can say that in symmetry we have an ordered repetition of form or measurement; though this does not imply that *any* such repetition, such as two equal squares, constitutes symmetry.

Consideration of geometrical symmetry in one dimension would be unrewarding here, and we therefore consider it first in two dimensions and then in three.

**6·5311. Symmetry of plane shapes.** A plane figure is symmetrical about a line in its plane if the reflection in that line of every point of the figure is itself a point of the figure; thus if the figure were folded along that line the two parts would coincide. The converse of course holds, and in geometry we normally deduce the existence of an axis of symmetry from the balance of the figure rather than vice versa; then having done so we can go on to say that what happens on one side must also happen on the other. Thus the balance of an isosceles triangle points to the existence of an axis of symmetry through the vertex, which belongs to both sides. We at once infer that the base angles are equal; the doubting can be asked 'Which would you expect to be bigger?'.

Again it could hardly be doubted that a circle, on account of its perfect balance, is symmetrical about any diameter; and it follows at once that the tangent at one end of the diameter makes equal angles, that is right angles, with the diameter (6·582 Fig. 27 iii).

Such symmetry, about a line, is sometimes known as T-symmetry.

Many children will be quick to feel that there is some kind of balance about a parallelogram although it has no axis of symmetry, and the simple idea that it can be turned through two right angles about its 'centre' and still look the same may lead to the concept of symmetry about a point, or S-symmetry. Here, the repetition of form after a half-turn implies the existence of a point which is such that the reflection in it of every point of the figure is itself a point of the figure (Fig. 13). At the school stage this kind of geometry is more interesting than mathematically important, for such figures are not common, and their main properties are probably known before the S-symmetry becomes apparent; but the artistic value is referred to later.

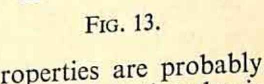


FIG. 13.

There is a modified form of S-symmetry where a turn through an angle other than 180° repeats the original pattern, as in such figures as those in Fig. 14.

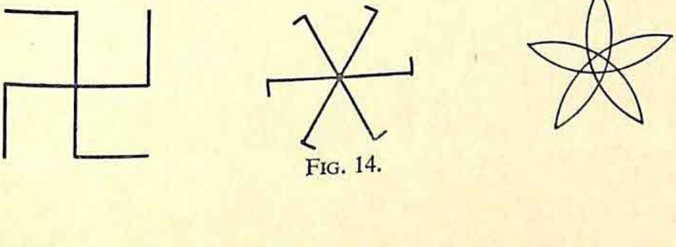


FIG. 14.

Discovery of these symmetries is a matter of discernment, not of applying any simple tests, and it is a good exercise to look for axes and points of symmetry in such figures as the equilateral triangle (3 axes), the square (4 axes), rectangle (2 axes), the figure-of-eight (2 axes), the ellipse and so on. An excellent opportunity is afforded for the pupils to suggest properties and to discuss their importance.

All regular polygons have such symmetries, and deductions concerning inscribed and circumscribed circles and the lengths of diagonals can be made. The circle is, of course, the perfectly symmetrical two-dimensional figure, being S-symmetrical about its centre and T-symmetrical about any diameter. This is the basic reason for the universality of the circular wheel.

**6·5312. Symmetry of three-dimensional figures.** Three-dimensional figures can have (i) symmetry about a plane, as a table drawer or a garden fork, (ii) symmetry about a line, as a two-bladed air-screw or the fitment which holds an electric lamp, or (iii) symmetry about a point, which is seldom met alone. There is also the modified form of symmetry about a line found, for example, in a three-legged stool and the cutting cylinder of a lawn mower. Many figures, of course, have more than one of these types. The sphere is the one perfectly symmetrical three-dimensional figure, being symmetrical about its centre and about any line and any plane through its centre. It is for this reason that so many games depend on a spherical ball.

**6·5313. Qualitative symmetry.** Besides repetition of length, *quantity*, there is repetition of *quality*. Thus in the second figure in 6·588 for establishing the relation between the angle at the centre of a circle and the angle at the circumference, there is qualitative symmetry on either side of the constructed diameter; what is true for one isosceles triangle is also true for the other.

Symmetry of form in algebra and trigonometry is most conveniently mentioned here since most pupils will not encounter many examples of it and it is of a qualitative kind. For example the expression $x^2+xy+y^2$ 'treats both the letters $x$ and $y$ alike' and is therefore said to be symmetrical in form.

Again if $a, b, c$, and $A, B, C$ be the sides and angles of a triangle any measure which is related symmetrically to the triangle, such as the area, must either be given by a symmetrical expression or by a triad of unsymmetrical expressions of common form. Thus the area is given either by the formula $\sqrt{s(s-a)(s-b)(s-c)}$, (which is a little outside our scope) or by $\frac{1}{2}bc \sin A$, which implies the existence also of $\frac{1}{2}ca \sin B$ and $\frac{1}{2}ab \sin C$ as equivalent formulae (6·62).

**6·5314. The occurrence of symmetry.** We are always meeting or using symmetry, and its beauty and usefulness can be appreciated without a need to analyse it being felt; the teacher should not force analysis on the pupil, but anyone wishing to build up a symmetrical pattern will find some knowledge of the principles helpful. Pupils will have cut out patterns in folded paper and then opened them out, e.g. to make open-work d'oyleys, or have used a pattern of half a yoke or half a collar on folded material;

they will probably have played the game of ' skeleton ' by writing a signature in ink along a line on paper and then folding the paper along the line while the ink was still wet. Many, wanting to make a drawing of a vase whose shape has pleased them, will instinctively pencil in a central line as a guide in getting the two sides alike; and some will welcome the help of Mathematics while others will feel that a freer if less accurate drawing better expresses what they want.

The mirror-image type of symmetry can be found in early art, for example in some Egyptian patterns, in the craftsman's jug or spear and the embroiderer's design; and qualitative symmetry appears for instance in early religious pictures, such as the triptych with the central figure supported on either side, and the balance of the sheep and the goats.

In architecture, S-symmetry of the two-fold and four-fold types is common. Five-fold symmetry, as well as other forms, is frequently found in the organic world as in flower-shapes but does not occur in the inorganic world of nature. Six-fold symmetry, however, is to be found in snow crystals, in the section of the honeycomb, and so on.

**6·5315. Asymmetry ; conclusion.** The recognition of asymmetry follows from the recognition of symmetry, and asymmetry has its own importance in theory and practice. It is of interest that it is the asymmetry, often slight, of the human face and hair that makes a photograph of oneself appear subtly different from what one sees in a mirror.

As suggested in the opening paragraph of this section, symmetry is something more than an aesthetic study of shapes. For instance it may form the basis for many intuitive generalisations about shapes. This idea is developed in 6·582, together with an extension to asymmetry. The loci determined by some conditions of equidistance are also examples of symmetry (6·556).

Again many important deductions depending on symmetry are made in applied Mathematics. For example questions of balance, centres of gravity and of pressure which occur in mechanics can be treated by the use of symmetry.

**6·532. Plane shapes bounded by straight lines.** In 6·53 the prevalence of the rectangle and the circle has been mentioned, but many shapes are also built up on the triangle. Pupils will almost certainly be familiar with the word *triangle* on entry to the Modern School, but they may associate it only with a complete shape and not with an outline made up of three lines. Almost certainly they will know best the right-angled triangle, as half a rectangle, or the triangle which has symmetry, like the gable end of a house. If they call this an *isosceles* triangle there will be no special emphasis in their minds on the fact that two sides are equal; it will just be a general way of recognising the symmetry. Small tiles which are equilateral triangles will be seen to have this symmetry whichever side is used as base; this again will seem their characteristic property, rather than the equality of the three sides. The same shape may be recognised in some printed design or in the print left in soft soil by a three-legged stool.

Ruling straight lines in order to form a triangle should at first be for a purpose, for example, because the pupil wants to make an exact drawing to serve as part of a model, or to make a scale drawing to solve some simple problem of surveying. The number of measurements to be made in order to copy an existing triangle will be considered in the next section on congruence, but once again we would stress that the logical arrangement is for the convenience of the teacher, and that the pupils' development is unlikely to follow a strict logical order.

Of the rectilineal figures built up from triangles, pupils are likely to become familiar with the quadrilateral, and its particular forms, the trapezium, parallelogram, and rhombus, and to learn some of their properties.

Among polygons those which are regular are likely to be noticed first, and will form the basis of much pleasurable pattern making, particularly the regular hexagon and the regular octagon. These two can be made of tile mosaics, equilateral triangles for the hexagon, and right-angled triangles and rectangles for the octagon. When inscribed in a circle they can be accurately completed without the use of a protractor. The dodecagon formed by drawing radii to bisect each side of a hexagon and then joining the twelve points on the circumference is a comparatively easy next step. For the pentagon and decagon it will be best to begin with the angles at the centre.

**6·533. Prisms.** Through experience of various containers, pupils will be familiar with square, triangular, hexagonal and possibly other *right prisms*, i.e. those solids which have base and top identical in shape and size and have rectangles as their side faces. The right cylinder will readily be taken as a prism on a circular base; the fact that its curved surface opens out into a rectangle is a further link with prisms. These solids can be pictured as made up of a pile of tiles of the same shape as the base. If the tiles overlap instead of fitting, so that the pile leans over, they give a suggestion of an *oblique prism*, i.e. one whose side faces are inclined to the base at an angle not a right angle and which are not in general rectangles. An oblique cylinder will be recognised as one whose circular top is not directly over its base.

Finding the height of prisms, i.e. the perpendicular distance between the planes in which lie the base and the top, is an interesting practical exercise. The analogy with piles of tiles leads to a realisation that the volume is given by multiplying the area of the base by the height.

**6·534. Pyramids.** Pyramids occur less frequently than prisms in ordinary life but pupils are likely to have seen a square pyramid surmounting a square column and will certainly have seen cones. These solids will come to be recognised as having bases like those of prisms but having a point or vertex to which the edges of the base are joined. Right pyramids or cones are those in which the vertex lies on the perpendicular to the base drawn from its centre. The side faces of pyramids are triangles and finding their area is an interesting exercise, especially if it is compared with the area of the rectangular face of the corresponding prism. If a pyramid is given theoretically in terms of its base and height a section will have to be pictured and drawn to determine the slant height, i.e. the height of the triangular face. The curved surface of a right circular cone offers a stimulating problem to abler pupils who can work out the theory from the practical procedure of ' developing ' it into a sector of a circle (6·574).

Geometrical methods of dealing with the volumes of pyramids and cones are more difficult than for prisms. A relevant fact is that 6 right square pyramids whose height is half the side of the base can be fitted together to form a cube, or, to look at this from another point of view, that a cube could be divided into 6 square pyramids each with a cube face for its base and the centre of the cube as vertex. A model can show this structure and lead to the realisation of the one-third relationship with the volume of the corresponding prism. A hollow cone can also be used to fill the corresponding cylinder to reveal that the ratio 1 : 3 appears again as the ratio of the volumes (6·537).

**6·535. The Circle.** It will no doubt be generally accepted that the store of general knowledge desirable for the ordinary person would be incomplete without consideration of the circle and its measurement. The names of the parts of the circle, diameter, circumference, radius, sector, arc etc. are in common speech and by discussion and example (6·506) their meaning can be clarified. The length of diameter provides the standard of size for domestic utensils (cake-tins etc.), guns, engine parts and bicycle wheels; whereas the circumference is concerned in a circular running track, the fence round a pond, the girth of a tree and, approximately, the path of the earth around the sun (6·562).

Much Mathematics may be done without any occasion to compare the radius of the circle (or the diameter) with the circumference; the occasion often has to be contrived. The need to find the area may arise earlier through interest in the capacity of a cylindrical can. Probably the most vivid encounter with the idea of the length of the circumference occurs in considering the working of a winch, with its obvious connection between the raising of a load and the length of rope wound on to the drum. Another basis of discussion, but more complicated, is the distance travelled by a bicycle for each turn of the pedals. Once discussion is aroused, experiment may be begun.

Although a close approximation to the value of $\pi$ is not within their power, pupils must have opportunity to realise that there is a number which is the same for any circle, large or small, and they must make some attempts to measure diameters and circumferences and to find this number.

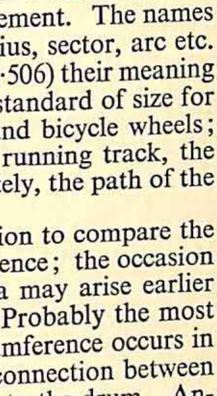


FIG. 15.

They may notice in a circle, with radius $r$ that the circumference is longer than the perimeter of the inscribed regular hexagon, so familiar in pattern drawing; that is the circumference is more than $6r$ since $AB = r$ (Fig. 15). If a square is drawn outside the circle, the circumference is seen to be less than $8r$. Thus $c$ lies between $6r$ and $8r$.

These observations have the advantage of drawing attention to the fact that the size of the circle does not matter. Different pupils should draw different sized circles and look at one another's drawings. Questions may follow, 'Can we get any nearer to $c$?', 'Is it more than $7r$?' and so on. This would be a good preliminary to attempts at measurement.

The teacher should refer to 6·82 on approximation in measurement. In schools much of the experimental measurement of the circumferences of circles is unpractical because the objects measured are so small that the error in measurement is large in comparison with the length measured. This will lead to serious discrepancies between pupil and pupil in comparing the circumference and diameter of a penny, but for a one gallon paint can the ratio $\frac{\text{circumference}}{\text{diameter}}$ will agree more closely. Error can be further reduced by the following methods:

*For circumference :*

(*a*) by wrapping several turns, instead of one turn, of thread round a cylinder and then dividing the length of thread by the number of turns.

(*b*) by wrapping a strip of paper round the cylinder and pricking through the two thicknesses with a pin, opening out the strip and then measuring the distance between the pricks.

(*c*) by marking a point on the circumference of a large hoop such as a bicycle wheel, rolling the wheel along a straight line and then measuring the distance covered in one revolution (or in several).

*For diameter :*

(*a*) by placing several identical objects side by side against a straight edge with a set square or other right-angled object at either end, then measuring the distance between the set squares and dividing by the number of objects.

(*b*) by measuring the diameter directly using calipers and diagonal scale or micrometer with vernier.

It is a good plan to organise the measuring in groups using different methods.

When several circular objects have been examined and their diameters and circumferences measured, the appropriate quotients can be found to one or two places of decimals. The various results should be compared and discussed. Their average can be found if desired, but when pupils have looked at the grouping of their answers they are usually ready to accept that a number exists which is the same for all circles. They can be told that the number is called $\pi$ (for perimeter) and that it has been calculated by mathematicians to a large number of decimal places; 3·14 is correct to two places of decimals; $3\frac{1}{7}$ is a good approximation. Calculations can then be used to find the circumference when radius or diameter is known and the radius when the circumference is known.

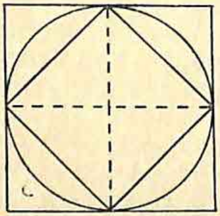


FIG. 16.

**6·536. The area of a circle.** When the area of the circle is considered an approach can be made through Fig. 16 which shows that it lies between $2r^2$ and $4r^2$. About $3r^2$ is then a possible guess, and pupils may themselves suggest that perhaps $\pi$ is coming in again. Circles can then be drawn on squared paper, estimates of the area made by counting squares, and the ratio $\frac{\text{area of circle}}{r^2}$ worked out.

A useful geometrical method is to cut out a number of equal circles and divide them into 4, 8, 16, 32, . . . sectors, and then to form them into the equivalent shapes shown in Fig. 17.

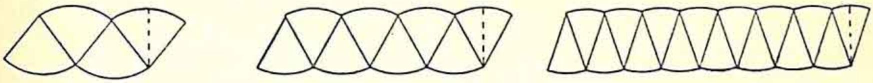


FIG. 17.

If the last sector is subdivided along the dotted line and half of it transferred to the other end, the shapes will be seen to become more and more nearly rectangles whose sides would be the radius and half the circumference. If the sectors could be small enough the circle would have the same area as a rectangle $\pi r$ by $r$ with area $\pi r^2$.

**6·537. Solids with circular sections.** Calculations about cylinders, cones and spheres also involve the use of $\pi$. When surfaces and volumes of prisms and pyramids have been studied, it will be possible to adapt the methods to cylinders and cones on circular bases. Cylindrical containers are so common that this knowledge is likely to be wanted. The methods for finding the formulae for the sphere are too difficult for these pupils but the formulae are sufficiently interesting and useful to be given from a reference book.

If suitable containers are available it is worthwhile to compare the capacities of a hemispherical bowl, a hollow cone, and a cylinder, all of radius $r$ and height $r$. They can also be compared with a block of 4 cubes of side $r$ (Fig. 18).

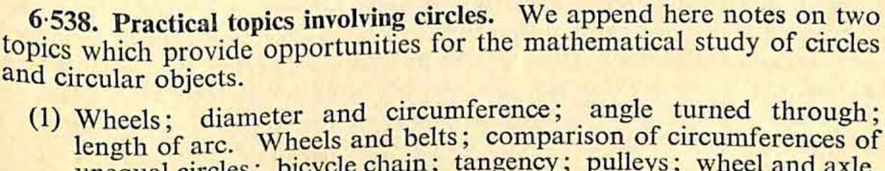


FIG. 18.

**6·538. Practical topics involving circles.** We append here notes on two topics which provide opportunities for the mathematical study of circles and circular objects.

(1) Wheels; diameter and circumference; angle turned through; length of arc. Wheels and belts; comparison of circumferences of unequal circles; bicycle chain; tangency; pulleys; wheel and axle. Gears; ratio of numbers of teeth. Revolutions; mileometer, speedometer. Graph of areas of concentric circles. The cylindrical wheel surface; volume; properties of shape of solid ring-tyres. Curve traced by a point on a rolling wheel (Appendix 1*b*).

(2) Earth Measurement. The study of latitude, longitude, and problems of navigation offers opportunities of great practical value and much interest; the difficulties are not formidable if ample use is made of diagrams, drawings on a sphere, wire models, etc. (7·213 to 7·215).

### 6·54. Similarity and Congruence.

The enlargement, or reduction, of a pattern, picture or map is an example of a practical problem which might present itself at almost any stage of a course. If the figure to be enlarged is a simple one the problem may be solved numerically by a change of scale and the fixing of a few lines and angles, but for more elaborate configurations, in which some freehand drawing may be necessary, different methods of enlargement offer interesting and profitable exercises.

The simplest method, probably, first superposes on the original a square mesh sufficiently fine to determine the position of enough salient points of intersection with the lines of the mesh; the mesh is then enlarged or reduced and the points corresponding to these intersections marked on it as a guide to drawing (Fig. 19).

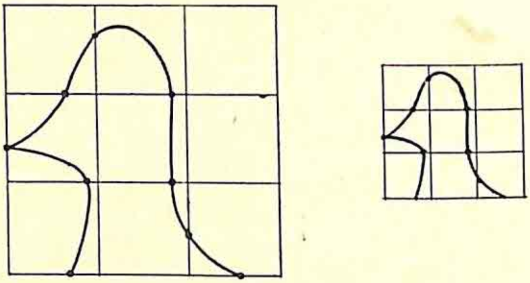


FIG. 19.

This method of construction and another one, conveniently referred to as 'the spider web method' (7·335 Fig. 69 i), are given in the section on drawing. Whether one uses an arbitrarily imposed mesh, or works from a centre of similarity, interested pupils will want to know 'why does it work?' and the practical activity will thus provide an opportunity for a study of the properties of shapes. Such a study will often be illumined when a distorted figure is produced through some error; it will be necessary for the teacher to help the pupils to clarify the meaning of the word *similar* as it is used with precision in mathematical language. Pupils may at first call all objects having the same 'shape name' e.g. rectangles, or pentagons, similar figures: they must discover that two four-sided figures, each with 4 right angles, may be of very different shapes, one being long and thin and the other almost square (Fig. 20). Again two extremely dissimilar 5-sided figures can be drawn even if all their corresponding angles are made equal (Fig. 21). Similarity in its geometrical sense means that two figures can be changed one into the other by enlargement or reduction, a process very familiar to viewers of TV and to readers of the experiences of Alice in Wonderland where she was in danger of drowning in a pool of her own tears.

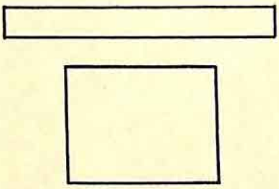


FIG. 20.

Pupils will also recognise the lack of similarity when they are so unfortunate as to have seats at the end of a front row at the cinema and find the actors unnaturally elongated.

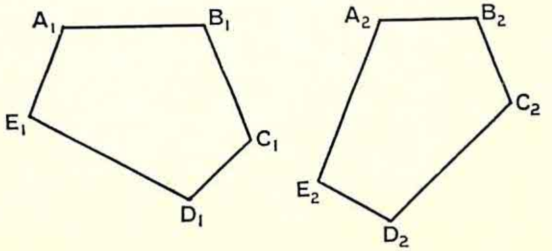


Fig. 21.

The idea that if two figures are similar one is a larger version of the other will thus be readily accepted, and pupils will admit that if the larger one is to be drawn the sides of the smaller must all be increased on the same scale. But if the figures are similarly placed and lines drawn mechanically with parallel rulers the fact that the interior angles of the figures are severally equal in pairs may escape notice. It is also much more difficult to appreciate that the ratios of the sides of one figure,

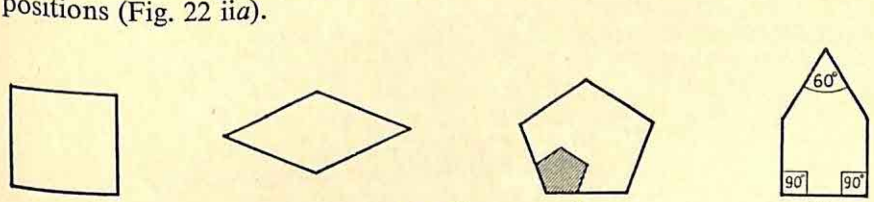
$$A_1B_1 : B_1C_1 : C_1D_1 : D_1E_1 : E_1A_1,$$

must be equal to the corresponding ratios in the other,

$$A_2B_2 : B_2C_2 : C_2D_2 : D_2E_2 : E_2A_2$$

if the figures are to be similar, though in fact this property is very useful in constructing scale models.

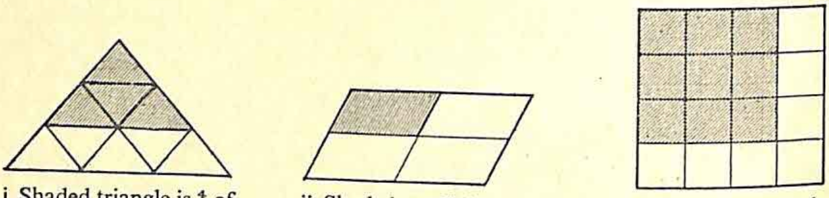
Attention can be drawn to the necessity of examining both sides and angles by comparing, say, a square with a rhombus, a rectangle with a parallelogram whose corresponding sides are equal, a *regular* pentagon with one which is equilateral but not equiangular. It is also instructive to cut out similar figures and to superpose one on the other in different positions (Fig. 22 ii*a*).



i*a* i*b* Fig. 22. ii*a* ii*b*

**6·541. Areas and volumes of similar figures.** The relationship between the *areas* of similar figures can also profitably be studied by means of cut-outs or diagrams; a few suggestions are given (Fig. 23).

The recurrence of the numbers 4, 9, 16, etc. (already known as square numbers) is more convincing than arguments about the 'squares on corresponding sides' but for the areas of irregular figures it is useful to notice that the square method of enlargement shows the dependence on the size of the squares used (Fig. 19).



i Shaded triangle is $\frac{4}{9}$ of the whole

ii Shaded parallelogram is $\frac{1}{4}$ of the whole

iii Shaded square is $\frac{9}{16}$ of the whole

FIG. 23.

The building up of 8, 27, 64 etc. cubes will be necessary to establish that volumes of similar solids are proportional to the cubes of corresponding sides. These similar solids may be bounded by curved as well as by plane surfaces; their volumes are then proportional to the cubes of any of their corresponding lengths; for example in a young and an adult animal there is an approximation to this. The rapid growth of surface area as lengths are increased and the still more rapid growth of volumes are well worth emphasising as of great practical significance.

Pupils may be led to discover that the similarity of two plane rectilineal figures depends fundamentally on the similarity of the triangles from which they can be constructed, the triangle being the only figure composed of straight lines which is determined uniquely by the measurements of its sides only. In order to survey a piece of land we first divide it into a number of triangles; to obtain rigidity in building frameworks the triangle is used by the introduction of tie-beams.

**6·542. Congruence.** As the pupil becomes aware of the property of similarity he will at the same time be developing the idea of the special case of similarity when the objects not only have the same shape but also the same size i.e. when they are *congruent*. This is a concept which, because of its very obviousness, is not easy for pupils to isolate. If a pupil wants to make a copy of a triangle which is there before him he will make a tracing, or prick through the paper at the corners and then rule lines, or devise some other way for himself; he is unlikely to try making measurements of the separate parts of the triangle unless this has been suggested to him by an adult. But he will be interested in the question whether, when certain measurements have been made in a field exercise and he and his friend make scale drawings from these measurements, they will both get the same answer to the problem in hand. If he has already had experience of similar triangles he will know that angles, even three of them, will not give any information about lengths, or not in units of length—if two angles are equal, then two sides will be equal, and if angles are unequal the larger will be opposite the longer side.

In the course of practical constructions pupils may attempt to draw a triangle from a variety of sets of measurements they have made (6·584), or to identify two triangles which have certain equalities. Their efforts will lead to discussion and some classification. Teachers should be prepared for the following possible cases of congruence; they include some overlaps.

(*a*) Two right-angled isosceles triangles with one side of one equal to the corresponding side of the other.
(*b*) Two isosceles triangles on equal bases and of equal heights.
(*c*) Two right-angled triangles with two sides of one equal to the corresponding sides of the other.
(*d*) Two triangles on equal bases, and the angles at both ends of one base equal respectively to those at the ends of the other base.
(*e*) Two sides of one triangle respectively equal to two sides of the other, and the angles between the two pairs of sides equal.
(*f*) Three sides of one triangle respectively equal to three sides of the other.
(*g*) Two obtuse-angled triangles with equal obtuse angles, and two sides of one respectively equal to the corresponding sides of the other.
(*h*) Two triangles with two sides of one respectively equal to two sides of the other, and the angle opposite the longer side the same in each triangle.

It is also desirable that teachers should consider the following types of case where triangles are *not* congruent.

(*a*) $ABC$ and $DEF$ $AB=DE$ $AC=DF$ $\angle BAC=\angle DEF$
(*b*) $ABC$ and $PQR$ $AB=QR$ $\angle ACB=\angle QRP$ $\angle ABC=\angle QPR$
(similar, but not generally congruent)

The following information is ambiguous, and the triangles may or may not be congruent.

$ABC$ and $XYZ$ $AB=XY$ $AC=XZ$ $\angle ACB=\angle XZY$
where $AB$ is less than $AC$

### 6·55. Loci and Envelopes.

Among the geometrical topics not usually included in the range of study for pupils under 15 is an investigation of loci and envelopes. Yet the idea of following or making a track or trail appeals to children from an early age: footsteps in the snow, the trail of a creature, the paper chase or even, rather differently, the wake of a ship as evidence of where it has been. Even at this quite informal stage the track can be described: ‘ The footsteps go from the back door to the right, then along the side of the house, across to the oak tree which they pass on the left and down to the nearest point of the stream etc. ’

**6·551. The locus of a moving point.** *The locus of a point*, viewed dynamically, is just such a track with the movements of the point which describes the locus regularised and the path determined by some guiding rule. Thus points at arm's length from a line at shoulder height are found by walking along the parallel line beneath it with arms outstretched, so that the fingertips trace out two parallel lines. Again a circle is drawn by a moving pencil point which is constrained by a loop of string kept taut round a pin, and an ellipse (Appendix 1*a*) by keeping the loop taut round two pins some distance apart.

A point on a moving object or on a deformable linkage of rods traces out a track or locus, for example in a recording barometer the point tracing the record moves in obedience to an up and down movement and a circular movement.

**6·552. A locus as satisfying a condition.** A locus may also be considered from a static point of view as an aggregate of points satisfying a condition. Thus, all the points whose distances from two fixed points have a constant sum lie on an ellipse. To find a place on a map 5 miles from *A*, I must realise that all points 5 miles from *A* are on the circle with *A* as centre and 5 miles as radius. It is this aspect of the locus that is used in fixing positions by loci (6·556).

**6·553. Loci in three dimensions.** From the static point of view a condition might be satisfied by the points on a surface in three dimensions instead of a path in two. Thus ' a given distance from a fixed point ' gives an aggregate of points forming the surface of a sphere and ' a given distance from a straight line ' the curved surface of a cylinder.

The loci discussed in 6·551 are limited to movement on a surface, but a track might be traced in three dimensions as for example the smoke-trail of an aircraft or the path of a hand on the handrail of a spiral stairway during a descent.

**6·554.** The idea of a locus may also be extended to the notion of a line tracing out a surface and of a surface tracing out a solid. For example the pencil point in a taut loop of string round a pin traces the circumference of a circle, the string itself sweeps out a circular disc, and if the disc were revolved about a diameter it would sweep out a sphere.

A high speed rotation as in an electric fan or the propeller of an aircraft shows the circular disc traced out by the radius as though it were a physical reality.

Again, a motion of translation instead of rotation might be used. For instance, one side of a rectangle moving parallel to itself and with one end remaining on the side at right angles sweeps out the rectangular area; this area might itself move at right angles to its plane to trace out a right prism.

**6·555.** Formal wording has been used in describing some of the loci above, but informal treatment of the subject will at first consist as far as possible of action, sketching and construction, together with the pupils' own descriptions which will only gradually approach any standardised form of words. Questions such as ' How does this revolving bob or rod travel? ', ' Why is this curious shape introduced into the sewing machine winder? ' (Appendix 1*c*) may at first produce answers which are not very shapely and which even if correct may not be complete.

Thus the question ' Where will the vertex *P* of an angle of a cardboard template lie, if it slides with the arms always touching two pins *A* and *B*? ' (Fig. 24) may be dealt with by marking a few points, including the one in which $AP=BP$, and observing that it looks like part of a circle. Further observations may lead to noting the positions in which $P$ approaches $A$ or $B$. The centre of the circle may be found and, as a final step, symmetry noted as giving another part-circle, the reflection of the first in $AB$. All this will lead to a great deal of informal statement; a formal summary may wait until the locus has been accepted and used. (See 6·588 for angles in the same segment of a circle.)

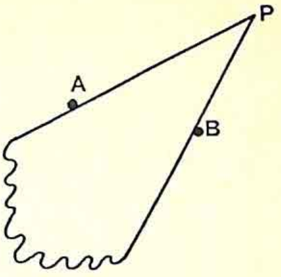


FIG. 24.

**6·556. Fixing position by loci.** The most familiar use of loci is for fixing the position of points. Problems of the 'treasure hunt' variety often involve finding the place where two loci intersect. The intersection of loci is implicit in giving the coordinates of a point. For instance, to take the less familiar polar coordinates, a point is fixed by saying

(i) it is on a line making a given angle with a fixed line through a point $O$, e.g. a line from $O$ bearing N.E.

(ii) it is a given distance from $O$, e.g. 5 miles. It therefore lies on a circle round $O$ with radius 5 miles.
The point is fixed by the intersection of the circle and the line bearing N.E. from $O$.

The following loci with their construction are worth study:
The locus of a point

(*a*) equidistant from two given points $A, B$: the perpendicular bisector of $AB$

(*b*) equidistant from two given lines: the *two* bisectors of the angles between the lines

(*c*) a given distance from a given line: two parallels either side the line (6·551).

These constructions can be thought of as having three dimensional extensions, thus

(*a*) becomes a plane through the mid point of $AB$, having $AB$ as perpendicular.

(*b*) becomes a plane bisecting the angle between two planes (6·573).

(*c*) becomes a circular cylinder surrounding the line.

There are in addition interesting loci associated with the circle, e.g. concentric circles which contain sets of points equidistant from the circumference of the given circle.

**6·557. Curves drawn as loci.** The tracing of loci can be easily and interestingly carried beyond the conventional instances of straight line and circle. Details of the construction as loci of several curves are described in Appendix 1.

Curves may either be plotted point by point according to geometrical conditions or traced by a point controlled by mechanical means, for example by a linkage of rods or a rolling wheel.

**6·558. Envelopes.** The tracing of a shape by tangent lines, *an envelope*, is a fascinating experience which may easily be given to quite young pupils either by drawing lines according to some agreed rule, or by stitching on a card so that threads represent lines.* For instance if points are marked off round the circumference of a circle so as to divide it into equal short portions and a chord joining two points of division is drawn, or stitched, equal chords may be drawn by joining further pairs in order. These will appear to envelop a smaller circle. This is a simple rule with a simple resulting envelope; the more complex the rule for constructing the lines the more complex the pattern.

The curves obtained by constructing envelopes may not be identified as shapes with particular names but their study provides a foundation of valuable experience of the property of tangency.

Other simple examples are the curve of pursuit, described in the Association's *Mathematics in Primary Schools*, the ladder slipping down a wall with ends on the wall and ground, and the ellipse, parabola and hyperbola as described in Appendix 1*a*.

For further discussion of loci and envelopes the reader is referred to the Association's *Teaching of Geometry, Second Report*.

### 6·56. Conic sections.

The circle is the only one of this family of curves which is traditionally included in the early study of geometry, but if the teacher has the necessary background and interest, and chooses a predominantly practical approach, pupils can be interested also in the ellipse, the parabola and the hyperbola. Generally it will be wisest to approach some members of the family as individuals before dealing with the property which gives them their name, but in the following notes the property that the curves are sections of a cone is put first. Other approaches are suggested but the fact that the curves which result *are*, as well as appear to be, of the same type is not demonstrated, nor are the notes exhaustive of possibilities.

**6·561.** The following can be associated with the occurrence of this family of curves as sections of a cone or *conic sections:*

(*a*) The elliptic appearance of a circular shape from an oblique view point.

(*b*) The edges of shadows cast by objects with a circular edge, for example a lamp shade, from a ' point ', or concentrated, source of light.

(*c*) The model of a solid right circular cone cut by plane sections or a wire model of a double cone in which are inserted plane sections. The hyperbola is a section by a plane cutting both branches of the double cone and the parabola by a plane which is parallel to one of the slant lines forming the cone.

**6·562.** Among additional occurrences of the individual curves which might be noticed and recorded are the following:

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* Pattern drawing by this means is described in *A Rhythmic Approach to Mathematics* Somervell (Philip). This little book has long been out of print in this country, but it has been reprinted in America.

(*a*) The ellipse occurs as an oblique section of a circular cylinder e.g. at a right-angled junction of pipes, as the orbit of the earth round the sun (with axes in the ratio 7001 : 7000) or of comets which recur at regular intervals, as the inner curve of a crescent moon; in some arches.

(*b*) The parabola occurs in the path of a projectile treated simply, in the profile of a jet of water, in the curve of a suspension bridge (but *not* in a clothes-line), as a graph in the curve of squares (6·47).

Parabolic cylinders occur in the reflectors of bar fires and in the non-reflector windows of some shops.

The constancy of shape in the parabola makes it very easy to recognise in the diverse situations in which it is found.

Paraboloids of revolution occur in reflectors, e.g (of lamps in lighthouses and searchlights to send out a parallel beam of light, or in a telescope to focus light) and in the surface of a revolving liquid.

(*c*) The hyperbola is associated with inverse variation. Its construction (Appendix 1*a*) brings out the characteristic property that it approaches two intersecting straight lines (asymptotes) more and more closely but never reaches them. A few comets move in hyperbolic orbits and so never return.

If the curve lies in the obtuse angle between the asymptotes the resulting curve will recall the cooling towers at power stations.

An outline model of a hyperboloid of revolution can easily be made by thin cords joining two parallel circles; the fact that straight pipes can be laid along such a surface thus becomes apparent.

**6·563.** Notes on the construction of all three conics as loci and as envelopes are given in Appendix 1*a*.

### 6·57. General Ideas in Three-dimensional Geometry.

Solid bodies have already been referred to both as shapes (6·533 and 6·534) and volumes (6·523) but there are other important ideas and facts about three-dimensional space which we gather together here, though, as with other mathematical ideas, we recommend that pupils' study of them should be in connection with their experience.

**6·571.** Lines in space will be visible not only as the short lengths of line which form the edges of solid bodies but also as railway lines or telephone wires, as kerbstone edges, or as a skyline; invisible lines will join one's eye to the top of a distant spire, or stretch out to the stars.

Straight lines on a plane either meet or are parallel, but when lines are observed in three dimensional space some of them clearly miss one another altogether without being parallel. Such straight lines are called *skew lines*. All lines on the ceiling will either be skew or parallel to lines on the floor. Or again a slant edge of a square pyramid and the far side of the base are skew. The word must be learned and then used again in varied observations.

Later enquiry by abler pupils may give meaning to the phrases 'the angle between two skew lines' and 'the perpendicular distance between skew lines' but formal work on these topics would not be suitable.

**6·572.** Familiarity with the word *plane* in its geometrical meaning will comprise a number of particular ideas and associations: for instance the terms inclined plane, vertical plane, the idea that a plane extends in all directions i.e. that it has no edge, that the plane face of a solid is only part of a plane.

Meaning must also be given to the idea of a line lying in a plane, e.g. the diagonal of the base of a square pyramid, and to the idea of a plane section of a solid. The fact that planes intersect in straight lines may be shown by folded paper or observed in the gable ends of houses.

Generally it is easier to visualise several planes if one of them, or a pair of parallel planes, is thought of as placed horizontally or vertically like the floor or walls of a room. For instance, in thinking of a right pyramid on a square base, if the base is pictured as horizontal it is not difficult to see from symmetry the angle which measures the dihedral angle between a slant face and the base; central lines drawn in the square base and in the triangle of the slant face are clearly at right angles to their line of intersection. It is however much harder to appreciate the angle between two *slant* faces for neither placing nor symmetry is as helpful in drawing the necessary perpendiculars at some convenient point in the edge along which they intersect. Practical solutions with large-size models are suggested.

**6·573.** Pupils can gain from experience, and quite informally, notions such as those summarised below. The list is not exhaustive.

(*a*) The idea of a plane itself: carpenter's test; planes determined by two lines which intersect or are parallel; plane containing a line and the idea of revolution about a line.
(*b*) Two planes meet in a straight line or are parallel.
(*c*) A plane meets parallel planes in parallel lines and if a line *l* is parallel to a plane *S*, a plane through *l* cuts the plane *S* in a line parallel to *l*.
(*d*) The meaning of the following terms:
  a line perpendicular to a plane, fixing an upright;
  projection on a plane of lines and areas;
  angle between a line and a plane;
  angle between two planes;
  angle of greatest slope of a plane;
  skew lines, the angle between them, the perpendicular distance between them.

**6·574. Developable solids.** Solids with plane faces and some with curved surfaces, e.g. the cone and the cylinder, can be constructed by paper modelling, or to put it geometrically their surfaces can be unfolded and opened out into a plane. The outline of their faces placed properly for refolding is called the *net* of the solid. Thus the net of a cube consists of 6 squares which may be joined edge to edge in a variety of ways, for example as a row of 4 squares with one square above and one below the row. This pattern might be thought of as the walls of a room with ceiling and floor.

The making of polyhedra from their nets is an attractive and instructive practical exercise. The nets required for regular prisms are easy to visualise and the difficulties involved are constructional only.

The relation between the curved surface of the right circular cylinder and the rectangle (with one side equal in length to the circumference of the base) from which it can be formed may give rise to useful discussion. The nets of pyramids are easily envisaged and drawn, but when pupils make an attempt to construct a cone, triangles may be tried instead of sectors; the discovery that the arc of the sector is equal in length to the circumference of the base is memorable.

**6·575.** The *regular polyhedra* can be built up from their nets. This is a subject in which many books deal with the constructional detail. The approach to it and the ideas which may be derived from it, apart from the interest aroused by the existence of such forms, should be carefully considered before it is decided to undertake a task which might easily become mechanical.

In the construction of any solid from its net the interesting fact should emerge that the sum of the angles of the faces of any *solid angle* must be less than 4 right angles.

**6·576. Solids of revolution.** The idea that a solid may be formed by the revolution of a plane shape, or its surface traced out by the revolution of a straight or curved line, has been discussed briefly in 6·554 and paraboloids and hyperboloids are mentioned in 6·562. The symmetrical properties of such shapes make them important in a machine age.

The cylinder and the cone are also solids of revolution traced out by the rectangle and the right-angled triangle respectively.

The sphere is the most important member of this family of solids and some study of its geometry is called for both because of its interest and because of its practical use in earth measurement (7·213, 7·214). If the earth is thought of as a solid of revolution it is easy to picture lines of longitude and latitude as great and small circles. Most pupils should appreciate what is meant by great and small circles of any sphere, angles at the centre of a sphere and at the centre of a small circle. The consideration of spherical triangles is not as difficult as might be supposed, especially if treated in connection with the distances between places on an actual globe.

### 6·58. Generalisation in Geometry.

We have pleaded for the study of geometry to be based firmly on spatial experience and we have pointed out how the pupil's ideas of space are clarified by giving them expression in speech, drawing and action. In this section we examine the place of intuition, of argument and of mere repetition of instances in leading a pupil to accept some of the more complex generalisations of geometry.

As a first step we divide general statements into two classes:

(i) those which are accepted by the pupil as generally true with little question.
(ii) those which are not readily discovered by the pupil himself without help from the teacher.

The first type may be accepted immediately from the wealth of past experience or the perception of the nature of the shape of the structure, e.g. the equality of the diagonals of a square. If acceptance is not immediate the teacher may be tempted to offer measurements as the grounds of belief. This needs careful consideration and is discussed in 6·581.

To establish the second type of statement the pupil may be guided to argue in short steps from facts already known, e.g. in discovering that each internal angle of a regular hexagon is 120 degrees; or, if the mathematical techniques are too difficult for him, measurement may be used to convince him of the *likelihood* of its truth. The implications of this latter method must be fully understood by the teacher and are so important that the next paragraph is devoted to them. (See also 6·535 on the discovery of $\pi$.)

**6·581. The argument from measurement.** We must point out at once that measurement, important as it is as a part of geometry, can never by itself constitute proof. By its nature it is approximate and particular. It can give a number of instances which make it *likely* that a certain statement is true and so give a clue which can be followed up. If the results of a number of measurements of a particular quantity are tabulated the probability of the truth of the statement may be seen to be high. But the truth has not been *logically* demonstrated; it has only been made acceptable as a working hypothesis. Sometimes, as in the determination of $\pi$, pupils need to be told that mathematicians can in fact give a complete proof.

The use of particular numbers in making statements is a different matter from quoting measurement as an argument. It may help the understanding of pupils to state an individual case thus: (*a*) a triangle has two sides each 2 inches long; if the angle opposite one of them is 40°, the angle opposite the other will be 40°, or (*b*) if I know that two angles of a triangle are 40° and 60°, then the third angle must be 80° because the three add up to 180°. We now consider several instances of class (i), 6·58.

**6·582. The argument from symmetry.** The equal-sided triangle in (*a*) above is seen to have symmetry about the line bisecting the angle between the equal sides. At first, recognition of the symmetry may depend on the position of the triangle relative to the observer. Of the two positions shown in Fig. 25 the one with axis oblique may not appear to be symmetrical until the pupil has adjusted himself to such changes of position. He can then generalise through experience the property of equal angles.

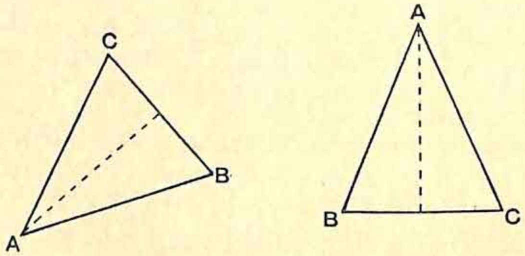

FIG. 25.

To demonstrate symmetry a cut-out triangle can be placed in different positions, or folded, or turned over and replaced in the hole from which it was cut. Some pupils would need to carry out these operations in imagination only; but, in any case, once symmetry had been perceived measurement would be unnecessary as a verification and would indeed be an anti-climax.

The converse idea that if the angles are equal the sides are equal may occur practically in some situation such as the following. From two points *A* and *B* on a line a distant object *C* is found to have bearings such that angle *CAB* = angle *ABC*; it will be readily agreed that *CA* = *CB*; *C* can be visualised intuitively and the line of symmetry recognised as the perpendicular bisector of *AB* (Fig. 26).

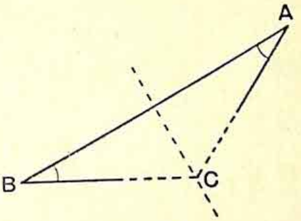

FIG. 26.

The property of the equal-sided triangle is made vivid by considering what happens if the sides are unequal. This will help to emphasise the important point that equal angles *always* go with equal sides, and unequal angles with unequal sides.

Properties involving perpendiculars can often be generalised from the perception of symmetry. If a pattern is pricked through a folded paper which is then opened out, the lines joining corresponding points are perpendicular to the axis of symmetry. Thus, it is possible to say that

(*a*) the diagonals of a rhombus are at right angles.
(*b*) the figure formed by joining the mid-points of the sides of a rectangle is a rhombus (Fig. 27 i).
(*c*) the triangle formed by joining alternate vertices of a regular hexagon is equilateral (Fig. 27 ii).
(*d*) the diameter drawn perpendicular to the tangent of a circle bisects all chords parallel to the tangent (Fig. 27 iii).

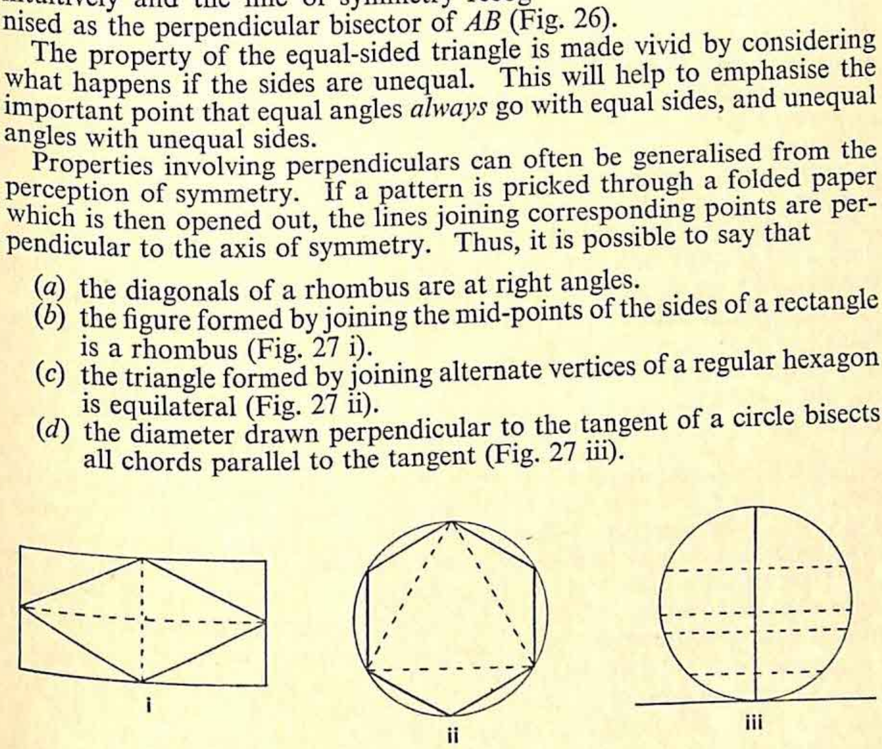

FIG. 27.

**6·583. The argument from orderliness.** Generalised statements can also be made on the grounds of the orderliness which parallel lines give to a configuration. In 6·513 the equality of alternate and corresponding angles made by parallel lines is treated in this way. Similarly if parallel lines 'are equally spaced', a non-technical phrase for 'cut off equal intercepts on a transversal at right angles to them', it is readily seen that they cut off equal intercepts on *any* line crossing them. A special case of this property, that the line drawn from the mid-point of the side of a triangle parallel to the base divides the other side in half, appears as an expected orderliness. The application of this property in the use of the diagonal scale reinforces its generality.

**6·584. The argument from the necessities of construction.** Another class of accepted generalisations is based on the necessities encountered in problems of construction. To eliminate the influence of symmetry we will consider drawing to scale an irregular 4-sided playing field. At a first attempt pupils may be content to measure the 4 sides, only to realise when they come to make the drawing that further information is necessary. At this point a demonstration with a framework of 4 jointed rods is likely to evoke the suggestion of choosing a diagonal as that fifth measure. This may lead to a realisation of the rigidity of a 3-rod frame and a discussion of the convenience of dividing shapes into triangles in order to reproduce them. From this will arise questions about the information required to construct a triangle of particular shape and size. All possible cases of congruence as listed in 6·542 could now be dealt with as sets of measurements and such questions asked as ' Given this set of measurements can I copy the original triangle? ', ' Is a triangle determined by 3 angles? '. Very valuable ideas will be gained about conditions which are sufficient, redundant, incompatible, or ambiguous.

Pupils who can take part successfully in such discussions may yet be unable to identify and use the congruence of two triangles as part of a generalised logical argument but they will have understood the importance of triangular structures and will make use of them in countless practical ways.

If construction is taken to include movement, manipulation, and the making of 3-dimensional models, generalisations connected with loci, envelopes and the properties of solid figures can be established.

**6·585. Generalisations of type (ii).** The generalisations which fall within the class described in 6·58 (ii) will usually come as a surprise to pupils. They will need therefore to reach them either by some very convincing experience or else by argument based on facts already known. Truths of this type can often be seen in general terms and as implicit in the configuration being studied, if they are demonstrated by folding or by dissection and a re-assembling of the parts, e.g. the area theorems relating to parallelograms and triangles. Where, however, a few ordered steps of thinking can relate known truths to one another and so establish a new truth, the pupils find the intellectual experience a valuable achievement. The form of expression may not be the generalised statements of systematic geometry; it may make use of hypothetical numbers to illustrate the general case (6·581) but it must be shaped in the mind of the pupil as a truth which he could apply to *all* cases.

Three important generalisations of this class (6·58 ii) are discussed in the paragraphs which follow.

**6·586. The angle-sum of a triangle.** The fact that the angle-sum of a triangle is *always* two right angles is not likely to be discovered by the pupil for himself. It is not impossible that in watching, say, a model car on a triangular or other closed circuit (not re-entrant) he has seen that in making the circuit it has made a complete turn but the identification of the angles turned through and their connection with the *interior* angles of the figure of the circuit are not easy to see.

Again, the pupil may notice the sum of the angles of a particular triangle such as the half-square or the half-rectangle if he is encouraged to do so. It may then be convenient to begin with right-angled triangles and show by folding as in Fig. 28 that the two smaller angles together fit on the right angle. (Incidentally this folding shows also that the mid-point of the hypotenuse is equidistant from the three vertices.)

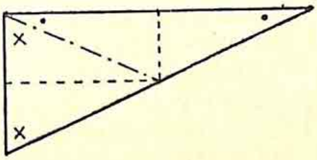

FIG. 28.

For the general triangle it is convincing to fold all three angles down to meet on the base at *P* (Fig. 29).

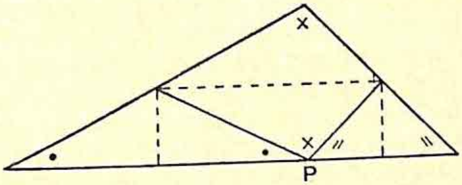

FIG. 29.

Alternatively, though less convincingly, the three angles of a paper triangle may be torn off and fitted side by side to form a ' straight angle '. As soon as the general truth has been accepted numerical work on the angles of a triangle should proceed, for not all modern school pupils will be able to pass on to the stage of showing that this property is a consequence of facts already accepted.

For those pupils who are ready for logical argument the familiar proof of the theorem, involving the construction of a parallel to one of the sides, may first be approached numerically. Fig. 30 shows a less usual position of the parallel in which the numerical values of the angles are easily identified by the pupil.

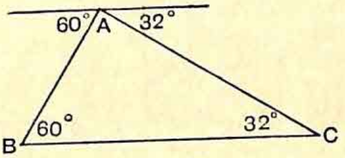

FIG. 30.

Extension to the sum of the angles of figures with more than three sides should in general be numerical only. The formula for the *n*-sided figure is likely to be reached only by the ablest pupils.

It will be noticed that measurement of the angles has not been recommended as a method of approach but once the angle-sum has been accepted as 180° calculation may well be used as a check on accuracy in drawing and measuring.

**6·587. Theorem of Pythagoras.** The unexpected relationship between the squares on the sides of a right-angled triangle is so useful in the elementary stages as a calculating aid that a numerical approach seems appropriate. Although measurement is known to be inconclusive, if right-angled triangles are drawn in which all three sides can be expressed as integral numbers of units (inches, centimetres, tenths of an inch, etc.) the error in measuring the hypotenuse is likely to be small and the results of a number of measurings will probably be closely grouped about the integer. For this work the following pairs of numbers are suggested for the sides containing the right angle: 3, 4; 5, 12; 7, 24; 6, 8; 10, 24.

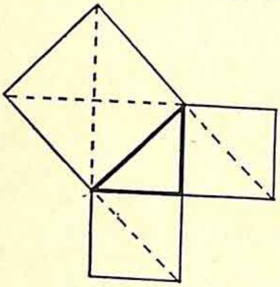

FIG. 31.

These constructions can be followed by some simple method of dissection selected from the many methods available. The special case of the isosceles right-angled triangle makes a good introduction. Figure 31 shows that the square on the hypotenuse can be divided into four triangles each equal to the original, and each of the squares on the other sides into two such triangles. Many pupils would be convinced by this drawing without need to dissect and superpose.

Figure 32 shows a general method based on two different dissections of the square on the sum of the two sides $a$, $b$ which enclose the right angle. If the construction shows the given triangle with squares actually on its two sides, as in the figure, this dissection is not difficult to understand. In each figure four triangles identical with the given one are needed to make up the square on $a+b$. This can be verified by superposition of the triangles.

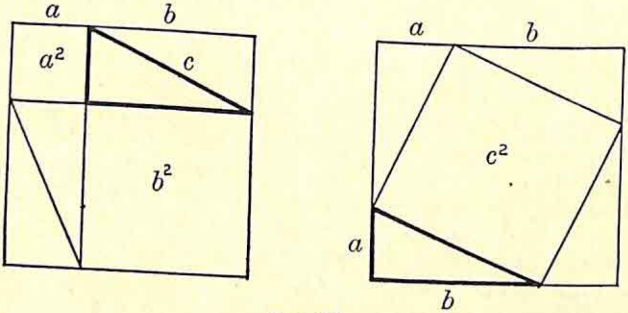

FIG. 32.

There are many ways of treating this theorem by argument but they are beyond most modern school pupils. A film strip is available summarising methods of dealing with Pythagoras's theorem; this would interest teachers but it is too full for these pupils.

**6·588. Angles in the same segment of a circle.** The remarkable fact of the constancy of the angle in a segment may first be discovered in the special case of the right angle in a semi-circle. This might have occurred incidentally as suggested in 6·586, folding having shown that the three vertices of a right-angled triangle are equidistant from the mid-point of the hypotenuse; or an examination of the angles of a triangle inscribed in a semicircle might have led to the pairing of the parts of the angles formed when the free vertex is joined to the centre. Figure 33 shows the numerical treatment: 'What is the size of the angles marked with dots?'

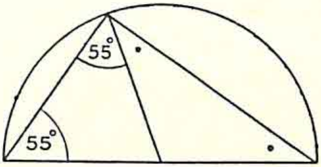

FIG. 33.

The temptation to deal with the general case by measurement should be resisted. A logical treatment of the relation of the angle at the centre to the angle at the circumference standing on the same arc is not too difficult if numerical values are given to the angles as in Fig. 34.

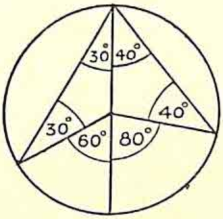

FIG. 34.

The use of the angle properties of isosceles triangles in these two cases is a good example of simple deductions from facts already accepted.

The theorem on the angle in a segment could be suggested by plotting the locus of the vertex of a triangle with a given base and a given vertical angle (6·555). An interesting application arises in navigation: when three known points, *A*, *B* and *C*, on a chart can be sighted from a ship, *S*, the position of *S* is determined by the intersection of arcs containing the angles *ASB* and *BSC*. In practice the arcs are not drawn but arms *SP*, *SQ*, and *SR*, are set to contain the correct angles and adjusted in position to pass through *A*, *B* and *C* respectively on the chart.

### 6·59. Some General Considerations.

We have used some of the traditional theorems of geometry to illustrate the way in which young people satisfy themselves of the truth of geometrical statements and proceed from acceptance of the particular case to recognition of a general principle. There is, however, plenty of opportunity to use such methods in dealing with less formalised spatial properties. Moreover it has been suggested in this section that the method of deliberate deduction, as opposed to intuitive acceptance, belongs to the later stages and is suited in general to abler pupils. Nevertheless it must be emphasised that connections between one fact and another can be established in quite informal ways *throughout* the course, so that conviction about a new fact can be reached by very simple reasoning by all pupils. Thus, observation of the behaviour of two rods jointed together at their mid-points can lead to the realisation that equal rods will always have their ends at points which form a rectangle, whereas unequal rods will define a parallelogram that is not right-angled (Fig. 35). Discussion of such simple linkages can be used to establish the properties of various 4-sided figures and their diagonals. Experiment can be made by varying the point of intersection of the rods.

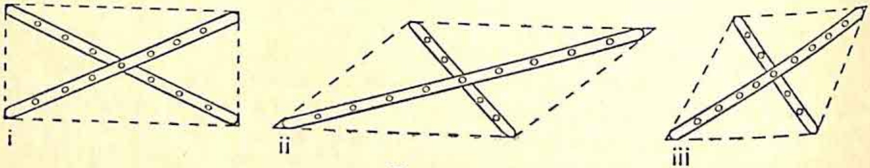

FIG. 35.

**6·591. The risks of intuition.** When experience brings to a pupil the certainty that some relationship is a universal fact he may be quite unable to give any reason for his conclusion except that ' it must be so '. It is unwise to press him to state his reasons until accumulated experience of talking about his discoveries enables him to convey a little of his thinking. Confident *use* of accepted properties or an explicit contrasting of one situation with another will usually enable a teacher to judge the extent of the pupil's perception.

It sometimes happens that a pupil jumps to conclusions which are wrong or only correct in particular cases. These occasions are valuable opportunities for discussion which will lead pupils to become critical of their spatial intuitions. They should naturally provoke further observation and experiment.

**6·592. Riders.** Geometrical problems, ' riders ', become possible as soon as a body of main facts is not only accepted but systematised. Riders in which angles and lengths are given numerical values, e.g. ' If a triangle has two equal sides and the angle between them is 70° find the other two angles ', are particularly suitable for pupils of average ability.

**6·593. Systematisation.** The kind of systematic arrangement of knowledge that is useful for modern school pupils is rather a gathering together of groups of related facts and properties than an attempt to establish a unified sequence. No particular order of such groupings should be prescribed nor is it desirable that the statements should be conventional. The individuals of a class may well summarise in their own way the knowledge which has a bearing on some problem or constructional task. Or they may collect together all the facts they know about a particular structure, e.g. a cylinder.

Discussion of the more fully systematised treatment of geometry suitable for abler pupils who may contemplate examinations in the subject is to be found in the reports of the Association referred to in 6·501.

## 6·6. Trigonometry.

*6·60. Relation to other mathematical topics.*  
*6·61. Introduction to tangent, sine and cosine.*  
*6·62. Range of work for average and for abler pupils.*  
*6·63. A problem for abler pupils.*  
*6·64. Construction of the sine graph ; the general angle.*  
*6·65. Using tables.*

### 6·60. Relation to Other Mathematical Topics.

Trigonometry is far from being an isolated mathematical topic; it has contacts with geometry, particularly the geometry of the triangle, with algebra, with graphical work and in its numerical form with arithmetic. It is indeed a most valuable study in which to display the unity of elementary Mathematics and it has the added advantage of linking up readily with practical topics, such as surveying, which are of interest to young people.

In trigonometry angles and lengths are brought together through the concept of ratio; but numerical trigonometry can be introduced so as to strengthen the concept of ratio rather than to depend upon it. Although trigonometry is not a difficult study, its early stages must be taken slowly in order that the pupil may gain familiarity with its new terminology, and facility in making statements in a new notation.

Details of the early mathematical treatment can be found in the Association's *Teaching of Trigonometry in Schools*, which should certainly be consulted for any work in advance of the solution of right-angled triangles, for example for alternative solutions to the problem in 6·63 below.

### 6·61. Introduction to the Tangent of an Angle.

Most of the work will be confined to right-angled triangles and it should be introduced as an outcome of situations which would otherwise have been treated by scale drawing and thus may arise in connection with problems concerning heights, distances, gradients and direction.

Although the practical situation chosen will determine whether one begins with the sine or tangent (or even cosine), whichever is first taken will give meaning to the new notation by its use; the formal definition of, for example, tan $A$ as a ratio will come later. Thus, from the measurement of the length of its shadow and the angle of elevation of the sun's rays, the height of an object can be found by scale drawing. Further experiments to find the lengths of shadows for different vertical poles of known heights will demonstrate the fact that the ratio $\frac{\text{length of pole}}{\text{length of shadow}}$ is constant, at of course one particular time. From this an unknown height can be found by a ratio calculation method exactly as Thales did with the height of the

Pyramid from the measurement of its shadow. Then, especially if the pupils are familiar with tabulations such as those for squares and logarithms, the table of tangents will not have an element of strangeness and can be shown to give the information that has been obtained heretofore from a scale drawing or from computation with ratios.

The use and examination of special cases such as *as high as it is distant*, with sighting along a 45° set-square, and *half as high as it is distant*, sighting along a specially constructed triangle may be sufficient at first to form a working basis.

**6·611. The sine.** An introduction using the sine rather than the tangent of an angle may be made by means of some problem which can be reduced to the geometrical form of knowing the hypotenuse and angles of a right-angled triangle and requiring another side. For instance the vertical *component* of a force known in magnitude and direction may have to be found, or the amount of *northing* after travelling on a known course, or again the height an object is raised when it has been pulled a known distance up a slope of given angle with the horizontal. It will be noticed that in these problems the sine of the angle, say $\sin B$ in the triangle $ABC$ whose hypotenuse is $a$, occurs as a fractional multiplier of the hypotenuse, that is in the form, side required, $b = a \sin B$ and *not* in the form $\sin B = \frac{b}{a}$. A pupil dealing with such a problem will fairly readily accept the essentials that $b$ depends on $a$ and on the angle; doubling, halving etc. the hypotenuse does the same for the other sides, and though $b$ is always shorter than the hypotenuse it increases if the angle $B$ increases. It is at this point that the table of sines can be shown to offer an alternative to drawing and measurement.

**6·612. The cosine.** The cosine of an angle can be approached in much the same way except that the side adjacent to the angle decreases as the angle increases. A useful example of this would be met with in finding the ground area covered by a sloping rectangular roof, where it would be natural to consider the plan of the roof as dependent on the angle which the roof makes with the horizontal.

## 6·62. Range of Work.

It is desirable that the majority of modern school pupils shall become acquainted with the meaning, determination and use of the sine, cosine and tangent of acute angles and with their graphs, but for many the work may proceed little beyond the use of tables for dealing with a right-angled triangle, and a growing familiarity with an important notation. From this work the usefulness of trigonometry as a practical tool will be apparent to pupils of ordinary ability, and there will be the added attraction of a novel notation and the ease of its associated computation, although as this usually involves decimal fractions, facility in understanding and manipulating these is a pre-requisite.

On the other hand, the fact that it is easy to give this subject real significance may make it a suitable medium for developing the mathematical range of rather abler pupils. To these trigonometry may arise during experiments in constructing triangles as an answer to the puzzling question as to *how* sides and angles are related; consideration of this may often bring real satisfaction to the pupil through appreciation of the power derived from generalisation. Again, a class interested in graphs, reaching the stage where the slope of a line is being considered, may begin trigonometry from the $m$ in $y=mx$, and consider the ratio in the first instance as a multiplier which is constant for a given slope.

The $a, b, c, A, B, C$ notation for the sides and angles of a triangle should be known long before trigonometry is broached. By choosing $c$ as the longest side of a triangle, even if $C$ is obtuse, there should be no difficulty in determining the height of the triangle as $a \,.\, \sin B$ or $b \,.\, \sin A$, and hence the area as $\frac{1}{2}ac \,.\, \sin B$ or $\frac{1}{2}bc \,.\, \sin A$, which again may lead into surveying exercises; or the need for such a formula might come from work in surveying.

Algebraic manipulation of ratios is, in general, outside the scope of the Modern School, although it may be worth while and quite possible to develop the statements $\tan A = \sin A/\cos A$, and $\sin^2 A + \cos^2 A = 1$ from the right-angled triangle.

### 6·63. A Problem for Abler Pupils.

For abler pupils the most difficult problem attempted trigonometrically might well be that of finding an inaccessible height sighted from two points in the same vertical plane a known distance apart.

In Figure 36 $BC = h \cot x$

$BC = h \cot y - d$, where the values may all be given numerically and cotangent need not be used unless tables of cotangents are available.

This same problem may also provide a target of difficulty in the solution of equations, for when the angles $x$ and $y$ are measured, and $d$ is known, the above equations, after the use of tables, become the simultaneous pair

$$BC = nh;\; BC = mh - d \quad (6{\cdot}361,\; 6{\cdot}60,\; 7{\cdot}135).$$

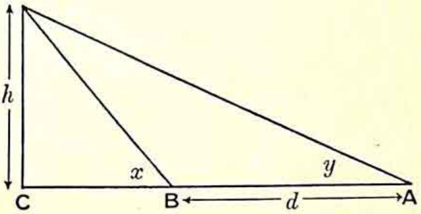

FIG. 36.

### 6·64. The Sine Graph and the General Angle.

With a background of good graphical work a class will be able to construct graphs of the trigonometrical functions geometrically by projection from the 'rotating radius' diagram (Fig. 37).

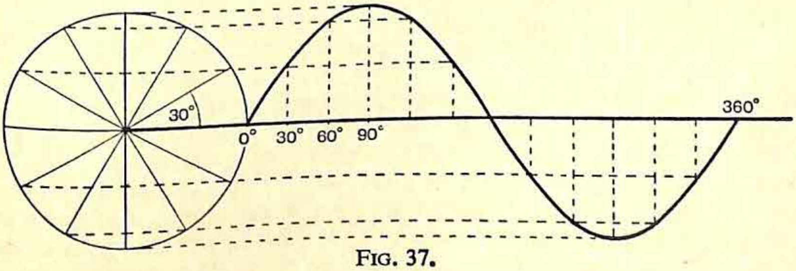

FIG. 37.

At first the sine of an angle will only have meaning for the angles of a right-angled triangle, that is in the figure from 0° to 90°; but pupils who have an interest in rotation will almost intuitively make the extension to the obtuse and then to the general angle. There is in the extension of this graph a wealth of mathematical ideas and everyday applications: the ideas of periodicity, function and negative number, the interpretation of the A.C. in 230 volts A.C., the meaning of cycle, frequency and wavelength, and even the reason for the symbol ∼ for cycle in 230 V., 50∼A.C.

### 6·65. Using Tables.

In addition to the possible developments suggested in 6·62, the abler pupils may be able to proceed to some work in the solution of the general plane triangle—and even be introduced to the spherical triangle.

These pupils often develop an enthusiasm for working with tables and need guidance on the questions of accuracy involved. For instance it is unsound to expect four figure accuracy in calculations based on the pupil's own measurements (6·82, 6·84, 6·85). The question of *how* values in the table are calculated may arise, in particular with reference to four-figure tables and if the teacher has a set of seven-figure tables handy to provide a stimulus, this may prompt more enquiry. The teacher may judge that the time is then opportune to produce the surprising fact that the trigonometric ratios are computed from series. If the class is equal to the shock, two valuable results may follow. The first, arising from the meaning of the series, is an explanation of the circular measure of an angle. The second is a new realisation of the depth and power of Mathematics.

## 6·7. Reckoning with Money.

- *6·70. General considerations relating to reckoning with money.*
- *6·71. The preliminaries to money computation ; addition and subtraction.*
- *6·72. Multiplication of £ s. d.*
- *6·73. Division of £ s. d.*
- *6·74. Multiplication and division using fractions.*

### 6·70. Considerations relating to Reckoning with Money.

**The performance with speed and accuracy of commonly used money calculations is a skill which is important** in everyday life, and which has a value in the labour market; but man's interest in money is social and utilitarian and his mathematical development could well take place entirely without it. **The reckoning involved is only an elaboration for a particular purpose of the fundamentals of reckoning discussed in 6·2, and by far the most important part of it is the reckoning done without the help of pencil and paper.** The tendency in the past to think of arithmetic as computation only has led to the treatment of money calculations as new rules, whereas the only special feature is an arbitrary connection between the values of coins used.

In this country money units are not decimally related, so that in passing from pence to shillings and from shillings to pounds we have to deal with scales of 12 and 20. The patterns of reckoning adopted for dealing with these changes of scale are similar to those for dealing with non-decimal systems of measures, which may account for the traditional association in arithmetic of money with measures.

Here however we deal separately with money and with measures (6·8), because of the fundamental differences between them. For example, a change to a decimally related coinage would involve less upset to people's established sense of values than a similar change in a table of measures. Money value is not stable; it is only the relative value of coins that is fixed. On the other hand a yard and an ounce are not only fixed in relation to a foot and a pound respectively, but each is an actual and traditionally established measure of quantity.

### 6·71. The Preliminaries to Money Computation.

The average pupil gains satisfaction and pleasure from being able to do correctly calculations involving money, and gains additional pleasure and a better understanding when money calculations are actually used in real situations (3·1).

We consider that hitherto time has been spent unprofitably on teaching techniques at too early a stage, in particular those for multiplication and division of amounts of money involving pounds, shillings and pence (4·61).

The four rules applied to money are now discussed below. The points already made in 6·23 about consistency of methods should be noted again here.

**6·711.** The essentials preliminary to all computation with money are:

(*a*) adequate knowledge of the four rules applied to numbers.

(*b*) understanding how to change from one unit to another, e.g. from shillings to pence and from pence to shillings.

**6·712. Addition and subtraction of £ s. d.** Addition and subtraction of money have been fully considered in the Association's *Mathematics in Primary Schools* (3·6), but it should be emphasised here that accuracy in this field is helped considerably by

(*a*) a systematic line of approach, i.e. each operation to be done in turn and in the same order.

(*b*) insistence on neat and orderly layout.

The widespread use of machines for office calculations has been discussed in 1·21, but we may add here that wherever machines are used there must also be people who know what calculations have to be done and can sometimes do them quickly and accurately without the machine.

### 6·72. Multiplication of £ s. d.

Application of the principles suggested in 6·711 leads to the following sequence

(*a*) multiplication of small amounts which can be dealt with mentally;

(*b*) the use and construction of ready reckoners;

(*c*) multiplication of larger sums by general methods requiring the use of pencil and paper.

**6·721.** Some special devices simplify and widen the range of multiplications which can be performed mentally by pupils having the requisite mathematical power. It is important however that such devices shall not be forced upon those for whom a short cut may not effect any saving of time or labour; for some pupils a so-called short cut may result in waste of time and even loss of confidence in more straightforward methods. The teacher may consider the following possibilities for some pupils.

1. *Multipliers allied to the money table.*

The finding of the cost of scores or dozens given the cost of one, i.e. using multipliers 20 or 12, can often be performed mentally. Thus for the scores rule

$$\begin{aligned}6\text{ shillings}\times 20 &= £6\\ \tfrac{1}{2}\text{ shilling}\times 20 &= £\tfrac{1}{2}\\ x\text{ shillings}\times 20 &= £x\end{aligned}$$

and for the dozens rule

$$\begin{aligned}6\text{ pence}\times 12 &= 6\text{ shillings}\\ \tfrac{1}{2}\text{ penny}\times 12 &= \tfrac{1}{2}\text{ shilling}\\ x\text{ pence}\times 12 &= x\text{ shillings}\end{aligned}$$

2. *Sums of money which differ from a ' round sum ' by small amounts*

Most pupils will be able to deal mentally by sum or difference methods with such amounts as

(*a*) 1s. $11\frac{1}{2}$d. × 16 = 32s. − 16 halfpence.
(*b*) 2s. $1\frac{1}{2}$d. × 16 = 32s. + 16 three-halfpences.
(*c*) 19s. 11d. × 43 = £43 − 43 pence.
(*d*) £1 2s. 6d. × 43 = £43 + 43 half crowns.

3. *Sums of money involving easy fractions of* £1 *or* 1*s.*

Such sums can often be dealt with mentally, but it must be borne in mind that modern school pupils are often confused by too early a use of fractions. Thus 12s. 6d. × 7 can be treated as 84s. + 3s. 6d., or as 5 half-crowns × 7, or, by those who have reached this stage in fractional work, as $£\frac{5}{8}\times 7$. Again 1s. 3d. × 43 is 43 shillings + $10\frac{3}{4}$ shillings.

**6·722. The use and construction of tabulated ready reckoners.** For calculations which have to be performed repeatedly as in calculating the cost of different amounts of commodities at stated prices, tabulated results of multiplication are much used in everyday life. The use and construction of tables form a useful step towards more general methods of multiplication.

Multipliers are split up into convenient parts and partial products added. A profitable discussion might well arise on the selection of a suitable range of numbers to be used in a ready reckoner. The simplest series is 1, 2, 4, 8, . . . etc. (as used in a set of English weights). From these any other desired number may be formed. Thus

$$\begin{aligned}3&=2+1\\ 5&=4+1\\ 6&=4+2\\ 7&=4+2+1,\text{ etc.}\end{aligned}$$

Alternatively, the reckoner might show subdivisions of the cost; £1, 10s., 5s., 1s., 6d., 2d., 1d.

The use and construction of such reckoners involves

(*a*) analysing numbers into suitable form (as above);
(*b*) analysing the cost and finding and using aliquot parts of money;
(*c*) using these separately;
(*d*) using them together;
(*e*) multiplying simple amounts of money by such multipliers as 2, 5, 10, etc., or using simple divisors in, for example, finding the cost at 6s. 8d. when the cost at £1 is known.

**6·723.** The construction of a ready reckoner constitutes a good introduction to any of the standard ways of performing multiplication of money, whether of the *practice* or the *wholesale* type; it closely resembles the former. These methods are set out in detail in most text books and in other reports of the Association. (*Teaching of Arithmetic*, 1932 ; *Mathematics in Primary Schools*, 1955.)

Practice methods make substantial demands on the judgment of the user, but abler or even average pupils may make a natural use of them in appropriate cases if their work has not been too rigidly standardised at the outset and mental work such as that described in 6·721 (2) and (3) has become habitual. Indeed practice methods can quite suitably be adopted as the standard method for abler pupils, but they are not recommended for the less able.

Other fractional methods are described in 6·74.

Less able pupils should from the first be encouraged to make a gradual approach to a wholesale method of setting down their work. For example 15s. 8d. × 32 may at first be set out as two separate multiplications: 15s. × 32 and 8d. × 32, which can later be set down side by side and combined thus:

| £ | s. | d. |
|---|---|---|
| | 15 | 8 |
| | | ×32 |
| 25 | 1 | 4 |
| | 21 | 12)256 |
| | 320 | |
| | 160 | |
| | 501 | |

The setting down only differs from two separate multiplications in combining the shillings before expressing them as pounds and shillings. It can be extended later as required to include pounds also in the multiplication.

Abler pupils will quickly appreciate that this is a way of setting down and combining three separate multiplications and of establishing a neat general method of procedure.

### 6·73. Division of £ s. d.

No general method should be taught as a ' rule '. Easy examples can be dealt with in their context by common sense. The idea of reducing a remainder of, say, pounds to shillings for further subdivision is a natural one if numbers are small enough for comprehension of the problem as a whole. Division can often be dealt with most appropriately by fractional or decimal methods, and for pupils whose number work is too restricted to use fractions the amount to be divided can at the worst be reduced to the unit of lowest denomination.

### 6·74. Multiplication and Division of Money using Fractions.

There are many occasions on which the use of vulgar or decimal fractions is the best way of dealing with money calculations.

**6·741.** Working may be much reduced by fractional methods if the shillings and pence are a simple vulgar fraction of £1, for example, 15s. × 52 = £$\frac{3}{4}$ × 52 = £39, whereas 16s. 5d. × 52 is not a subject for treatment by fractions since the numbers involved are too large and nullify the advantages of the fractional method.

A pupil may readily recognise the sum of money as a fraction of £1 but readiness to take the step to multiplication of fractions rather than of money is of slow growth and needs constant help and encouragement.

**6·742.** Many money calculations involve both multiplication and division and for these the substitution of fractional for direct methods is more generally labour saving. In their most economical form, however, these methods demand something more than the ability to express a sum of money fractionally in terms of the highest unit. For example there is an advance in thought in saying that the result of dividing by 4 and multiplying by 3 is the same as taking $\frac{3}{4}$ of the whole, or that in buying $\frac{3}{4}$ of some quantity I shall only expend $\frac{3}{4}$ of what the whole would cost me.

A more difficult example is the case where $\frac{3}{4}$ of £2 13s. 4d. becomes translated into £$(2\frac{2}{3} \times \frac{3}{4})$. Here *of* has been replaced by the multiplication sign, ×, and then shillings and pence have been converted into a fractional part of a £1.

Two examples are set out below to illustrate the various problems that may arise in working.

The emergence of a ' number rule ' from the consideration of the use of simple fractions in many contexts is dealt with in 6·246.

The teacher at least should realise clearly what he is asking of the pupil when making this transition.

*Example* 1. A man spends £2 5s. 6d. in a week on his petty expenses. How much is that for two days?

The following are different ways of stating what has to be done.

(*a*) Divide £2 5s. 6d. by 7 and then multiply the result by 2.
(*b*) (£2 5s. 6d. ÷ 7) × 2 = 6s. 6d. × 2 = 13s. *or* (£2 5s. 6d. × 2) ÷ 7.
(*c*) Find $\frac{2}{7}$ of £2 5s. 6d.
(*d*) $\frac{2}{7}$ of £$2\frac{11}{40}$.
(*e*) £$2\frac{11}{40} \times \frac{2}{7}$ (the transition from (*d*) to (*e*) is critical).

(*f*) $\dfrac{£91\times 2}{40\times 7}=\dfrac{£13}{20}=13\text{s.}$

(*g*) See 6·744 for decimal methods of working.

*Example* 2. Find 4 per cent of £230.

$$\begin{aligned} 4\%\text{ of }£230 &= \frac{4}{100}\text{ of }£230 \\ &= £230\times\frac{4}{100} \\ &= £23\times\frac{4}{10}\left(\text{or }£23\times\frac{2}{5}\right) \\ &= \frac{£92}{10}=£9\ 4\text{s.} \end{aligned}$$

*or* using decimals

$$4\%\text{ of }£230=\frac{£230\times 4}{100}=\frac{£920}{100}=£9{\cdot}2=£9\ 4\text{s.}$$

4% of £230 = £2·3 × 4 = £9·2 = £9 4s.; that is, divide mentally the sum of money by 100 by putting the decimal point in the correct place and *then* multiply by the percentage figure.

**6·743. The use of decimal fractions of £1.** Most pupils are reluctant to use decimal fractions, particularly where money is concerned. Two teaching errors are mainly responsible:

(*a*) the abstract approach to the decimal notation.
(*b*) lack of understanding of the occasions on which this notation is suitable and an advantage.

As an introduction to the use of decimals for English money the various decimal systems of coinage in Europe and the U.S.A. can usefully be discussed and specimens of coins shown. The history of their development and the derivations of their names form a profitable digression. The much discussed, and perhaps imminent, possibility of introducing a decimally related coinage into Britain leads readily to the fact that there are already some decimally related items in our currency.

In fact our coinage is very nearly a decimal system already for not only is the ten shilling note 0·1 of the five-pound note, but, and more usefully for our present purpose, 2 shillings = £0·1, and since there are 960 farthings to the pound, 1 farthing is a little more than £0·001.

Many different rules have been formulated for using these two facts to evaluate by inspection a decimal of £1 in shillings and pence, and to express a sum of money as a decimal of £1 correct to 3 places of decimals. It is doubtful if it is worth while for the ordinary pupil to learn these rules, but the following points deserve consideration:

(*a*) The exact expression of a sum of money as a terminating decimal of £1 is not always possible.

(*b*) If required to any specified degree of accuracy, money can be decimalised by the straightforward method of division thus:

Express £4 13s. $5\frac{1}{2}$d. as a decimal of £1.

$$\begin{aligned} \text{£4 13s. } 5\tfrac{1}{2}\text{d.} &= \text{£4 13s. 5·5d.} \\ &= \text{£4 13·45833s.} \\ &= \text{£4·672916} \\ &= \text{£4·6729 to 4 places.} \end{aligned}$$

Such a result might be required in a wages calculation to be fed into a machine.

(*c*) It is generally sufficient in evaluating a decimal of £1 in shillings and pence to deal with 3 places of decimals only, since £0·001 is less than 1 farthing and anything in the fourth place is smaller still.

It is also easy to deal at sight with the shillings and sixpences using the results

£0·1 = 2s.
£0·05 = 1s.
£0·025 = 6d.
and also perhaps £0·0125 = 3d.

**6·744.** Older and abler pupils in a Modern School may well come to appreciate the economy of arriving at an answer in decimal form in money calculations. For example, find the gross income to the nearest shilling if the income after removal of tax at 9s. in the pound is £230 13s.

$$\begin{aligned} \text{The gross income} &= \text{£230·65} \times \frac{20}{11} \\ &= \frac{\text{£4613·0}}{11} = \text{£419·363}\ldots \\ &= \text{£419 7s. to the nearest shilling.} \end{aligned}$$

Again in Example 1 (6·742) the use of decimals would be suitable not only for the sum of money given, but for one which did not work out neatly by fractional methods e.g. for £2 4s. 6d. as well as £2 5s. 6d., thus

$$\text{£2 5s. 6d.} \times \frac{2}{7} = \frac{\text{£2·275} \times 2}{7} = \frac{\text{£4·55}}{7} = \text{£0·65} = \text{13s. exactly.}$$

$$\text{£2 4s. 6d.} \times \frac{2}{7} = \frac{\text{£2·225} \times 2}{7} = \frac{\text{£4·45}}{7} = \text{£0·636 approx} = \text{12s. 9d. nearly.}$$

Direct methods are however often simplest (Example 1*b*, 6·742) unless the divisor includes tens-factors (Example 2, 6·742).

The dangers of using an inexact decimalisation if the amount is then to be increased by multiplication must be pointed out; the degrees of accuracy to be expected in different cases could be considered.

**6·745.** With pupils of fair ability considerable interest can be aroused in the different ways of performing a calculation, particularly during oral work where answers are obtained quickly. Accuracy and speed should however be the main aim, and method a secondary consideration. It is better for a pupil to use a rather longer method which he feels confident he can use accurately than a ' more intelligent ' method he does not fully trust.

With children of decidedly less than average ability, the aim should be to give one reliable straightforward method which can be used for both oral and written work. Short methods are likely to confuse the backward child. Here again the ability to work simple calculations *without* pencil and paper is of far greater value than the ability to work out more difficult calculations on paper. What is even more important for all pupils is that the work done in money computation should be related to its use in dealing with the economics of everyday life (3·1).

## 6·8. Measures.

*6·81. The range and type of quantities measured.*
*6·82. The nature of measuring : degrees of accuracy.*
*6·83. Reckoning with measures.*
*6·84. Accuracy in measurement as it affects computed results.*
*6·85. Reliability of computed results.*

In the secondary stage the range and type of quantities to be measured is greatly extended as the pupil's experience widens. The possibilities are discussed in the next paragraph, but what is included in the course must depend on the extension of practical experience both within and outside the mathematical course itself. Co-operation with other subjects is necessary (3·4).

The association of measuring with the development of ideas of number links this section with 6·2, in particular with fractions (6·151, 6·24, 6·247), irrational numbers such as $\pi$ (6·151 and 6·535, 6·536) and approximate calculation (6·27).

### 6·81. The Range and Type of Quantities Measured (1·23).

Measuring at the primary stage will generally have given entrants some acquaintance with at least the smaller commonly-used units for measuring length, weight, capacity and time (2·41*d*, 4·62) and also possibly of area. The measurement of area and volume is discussed fully elsewhere (4·8, 6·52). In what follows we discuss several categories under which other extensions of experience of measuring may be considered, but our suggestions are not exhaustive.

**6·811. Larger and smaller measures.** Extension of magnitude in the units of measure discussed and used may range widely from, for example, light-years at one extreme to wave-lengths of light at the other. Prefixes such as *kilo-* and *mega-* may arise; when the ratio of units exceeds, say, 1000 it is convenient to express any large numbers which are needed in standard form e.g. $6{\cdot}3 \times 10^8$ (see 6·247).

All courses will at least extend magnitudes to include clearer notions about such larger units as the ton and the mile and for smaller measures subdivisions of the inch.

In subdividing units for very fine gradings of size, decimal parts are generally used even where the larger measures, as in this country, are not decimally related. Thus in industry *thous*, meaning thousandths of an inch, are commonplace to the engineer, and per cent and per thousand are met widely in commerce and social statistics. Proposals that we should in this country introduce the metric system of units thus rest more on the argument for standardisation with other countries now using this system than on that for decimalisation which is already in use for small measures.

Pupils will not only enjoy making refinements in measuring for themselves (6·82) but they will also be interested in what can be done with modern equipment. For instance watch repairing shops are using an electronic type of watch testing instrument which can register time correct to a ten-millionth part of a second; this can be compared with the one-fifth second of the stop-watch.

**6·812. Measures of other countries.** Familiarity should be gained with the metric system of lengths, area, volume, capacity, and weight in contexts in which this system is already in use in this country and in pursuing an interest in the daily life of other nations (3·512, 5·331).

**6·813. Measures for particular purposes.** Particular measures may arise in the pupils' own activities for example in gardening (3·41), in athletics and games, in painting and decorating, in a pre-nursing course (3·51), etc.; or they may arise from the study of particular trades and occupations, for example, bushels of apples, crans of herring, stones of meat, reams of paper, vats, barrels, fathoms, cables, kilocycles etc.

**6·814. Compound units of measure.** The first compound unit to be considered will probably be a unit of speed (miles per hour) and the inverse variation of speed with time will need careful treatment. Other examples of compound units which may be used and analysed are other rates such as rate of fuel consumption in ships and cars (gallons per hour or miles per gallon), or the reverse in aeroplanes and army tanks (gallons per mile), or compound units of the product type, such as man-hours or kilowatt-hours.

**6·815. Units measured indirectly.** The nature of a measure need not necessarily be understood if the purpose and means of measuring it are clear. For example temperature may be measured by the height of a column of mercury, or electrical units used read from a dial.

Again the measure may be understood but the means of measuring it may be indirect as when change of length operates a rotating pointer or change of height is calibrated to register changes in volume. Calibration may be a school activity and a rain gauge may be both made and used.

**6·816. Measures of density.** Experiments in weighing at the primary stage will have led pupils to the conclusion that equal weights of different substances have different volumes or equal volumes different weights. This basic idea may require further investigation; for example it may be accepted for solids but the pupil may be surprised that the weight of a bottle of milk differs from the weight of the bottle filled with water. Some experience of the weights and volumes of typical substances must be amassed and a few standards such as the following established:

(*a*) A cubic foot of water weighs 1000 ounces, more than most schoolboys could lift. It occupies 50 pints, i.e. as much as the morning milk for 150 pupils, but the milk would weigh more.

(*b*) A litre of water weighs 1 kilogram, a litre of oil weighs less and a litre of milk more.

(*c*) The *density* of a substance is measured in ounces per cub. in. or grams per c.c. (mass per unit volume). Milk and most solids have higher densities than water; oil and cork have lower densities; so that, if water is chosen as the standard substance for comparison, the ratio expressing the relative density of milk is greater than 1 and that of oil less than 1.

(*d*) The density of mercury is 13·6 grams per c.c. A volume of mercury is 13·6 times as heavy as the same volume of water; the number 13·6 is sometimes called the *specific gravity* of mercury.

### 6·82. The Nature of Measuring : Degrees of Accuracy.

All measuring is to some extent an approximation, though the use of 'round figures' e.g. 2 hours, 3 lb., 2 c.c., 72° Fahrenheit, masks the fact and gives no precise information as to whether the 3 lb., for example, can be treated as 3·0, or 3·00, or 3·000, i.e. how accurate a balance would be needed to detect any error. Secondary school pupils must learn to appreciate this distinction between the exactitude which is possible when making a count, and the approximation which becomes necessary when making a measurement. An estimate of a measurement can be treated as a rough approximation to it.

If different groups of children measure the same playground they will get different results; discussion of the fact that they *always will* get different results, even after mistaken results have been rejected and all possible care taken, is a most important part of the teaching. It will often be worth while for them to watch the teacher performing the measurement twice and getting different results. The questions then arise 'Whose answer is right?', 'How can we know if our answer is right?', 'How can we work with measurements which are not exact?', 'Can anybody ever know the *exact* length of the playground?' and even 'Is there such a thing as an exact measurement?'. If these questions are well ventilated an understanding of approximation will have been based on a foundation of reality; the question 'How far can I rely on my measurements?' will have meaning, and the reliability of a result based on real, and hence approximate, data will be a matter for real and not merely academic consideration.

In Appendix 2 on statistics an example is worked out in which the measurements made by 26 boys of the chords of a regular hexagon of side 2 inches are examined statistically to give calculation of a mean value and a measure of deviation from the mean. Pupils may in fact in this way establish for themselves a standard of their performance within a group.

**6·821.** It is a mistake to postpone consideration of approximation until occasion arises for sub-dividing a tenth of an inch. If the suggestions of the last paragraph are accepted the pupil can become familiar with the idea of 'to the nearest foot' in measuring a playground or a garden, 'to the nearest yard' in measuring the length of a street or the dimensions of a field, 'to the nearest mile' in considering the length of a journey, 'to the nearest quarter-hour ' in considering an afternoon walk, ' to the nearest minute ' in timing his journey to school. When the pupil is familiar with the language and meaning of approximation, discussion of the reason why the nearest foot is appropriate in some cases and nearest yard or nearest inch in others should promote the more general ideas of degree of approximation, and percentage error (6·824). Out of this the abler pupils will ultimately gain some appreciation of the general rule that ' 3-figure accuracy ' is the accepted standard for direct measurement without the aid of special devices such as verniers or micrometers.

**6·822. Refinements in measuring.** A further step towards understanding degrees of approximation can be taken by considering the principles on which vernier scales and diagonal scales are constructed. Measurements made on a map of scale 6″ to the mile with a diagonal scale reading to $\frac{1}{100}''$ will give the true distances to within 3 yards, but such a map could not indicate the width of a footpath or a hedge. Similar comparisons can be devised with measurements of weight, or other quantities.

Pupils can also learn to read the dials of electric meters and other mechanically registered measurements. These will almost always be in decimals in terms of a single unit, and the number of figures to be read will determine the degree of accuracy.

**6·823. The purpose of a measurement.** Accuracy in measuring must be considered in relation to the use to be made of the measurement. If a classroom is to be measured, and then drawn to a scale of 1 inch to 4 feet, and the drawing ruler is marked in eighths of an inch it will be sufficient to make measurements of the room to the nearest half foot.

**6·824. Percentage error.** Another way of expressing degree of accuracy is by giving the percentage error. This may be too difficult an idea for many boys and girls, but calculation of it in simple particular cases may help to emphasise the idea that it is not the actual error but its relation to the whole measurement which is significant. In extreme cases this becomes self-evident; most pupils will agree that to make a mistake of an inch in giving the length of a finger is more serious than an inch in giving the length of an arm; but to give a convenient comparison, percentage error might be calculated.

Thus a mistake of an inch in measuring a page of a book might be an error of $12\frac{1}{2}\%$ (or 1 in 8), a mistake of an inch in measuring a girl's sleeve would be an error of about 1 in 18, or less than 6%; a mistake of a yard in measuring 100 yards would be an error of 1% but a mistake of a yard in measuring a mile would be an error of less than $\frac{1}{10}\%$.

### 6·83. Computation with Measures.

There is less excuse for labouring ' the four rules for measures ' of one kind and another than there is for elaborating and practising money rules. Reference must again be made to 6·23; the *only* fundamentals are the four rules for numbers of which any rules for measures are composed. Moreover it is unrealistic to give any measure in terms of a long list of different units, generally two will suffice. For example, if a distance is given as several miles a few feet more or less are not worth including and the only other measure mentioned may be yards and these should usually be given to the nearest hundred yards. Even with measures as near to one another in magnitude as pints, quarts, and gallons, it is probable that in practice gallons and parts of a gallon would be stated as gallons and *either* pints *or* quarts, or simply as gallons and a fraction of a gallon.

**6·831.** Computation with simple measures should include ready changing from one unit to another as required, but no elaborate ' reduction ', although converting miles per hour to feet per second is both reasonable and realistic.

Again some of the steps from a quite small to a large unit are worth finding and remembering. For instance 5280 feet to the mile is of interest in considering the scale of maps and there might be occasion to make use of 2240 pounds to the ton. If such numbers are used, the fact that they are the product of factors giving the steps between units more closely related should not be lost sight of; thus 2240 is 112 × 20.

**6·832.** If the fundamental processes are understood and the units themselves have meaning for the pupil, as recommended in the Association's *Mathematics in Primary Schools*, it is unlikely that a pupil will meet a realistic problem beyond his powers. For instance suppose he wishes to calculate the average weight of oarsmen in a rowing eight, he may have to add a number of stones and pounds and then divide by 8. It does not matter greatly in what form the sum is given provided that any number of stones remaining after division is brought to pounds and combined with the other pounds for division by 8. Thus, if the sum is given as 87 stones 71 pounds *instead of* 92 stones 1 pound, it is possible though clumsy to proceed with the division by 8. A pupil who knows what he is aiming at will probably achieve a result even if he obtains 10 stones 21⅛ pounds on the way.

**6·833.** In all the work in schools involving measures it is essential to bear constantly in mind the fact that only realistic measures should be involved and that they should be dealt with in a realistic manner. Too often in the past use has been made of measures to provide long and tedious calculations under the heading of ' Reduction ', and such problems as ' Convert 1 million inches into miles, furlongs, etc.' have produced nothing but frustration, boredom and a feeling of futility, and cannot be too heartily condemned.

**Realistic measures used in realistic calculations must be the keynote of teaching in the Modern School.**

Should any type of calculation be required repeatedly, the pattern of it can generally be determined by copying the pattern used for money units (6·712, 6·723 and *Mathematics in Primary Schools*).

### 6·84. Accuracy in Measurement as it affects Computed Results.

Elaborate work on approximations cannot be expected of modern school pupils, but the following points should be noticed:

(*a*) Approximation should always be related to *particular circumstances*, i.e. what is sensible or given about the measures, the accuracy of the instruments used, the size of the quantities involved.

(*b*) Respect for the number of significant figures should be cultivated. It is 2, 3, or 4 figure accuracy which is important (rather than the number of decimal places in relation to a unit); thus 0·023 in. is

23 'thous' which represents two figure accuracy although it is given to three places of decimals of an inch.

(*c*) Calculations may often decrease but never increase the number-of-figure accuracy. This is a most important point to be appreciated by the pupils.

Pupils of very modest ability may at least come to realise that the leading figures, i.e. the figures on the left, of both a measurement and a calculation which involves the measurement are more important and more likely to be reliable than those on the right. If examples are kept in touch with experience no loss of confidence in the accuracy of the computation itself need result.

### 6·85. Reliability of Computed Results.

It is important that the distinction between accuracy and reliability be clearly grasped. A computation should always be accurately performed, but the result may not be entirely reliable: it can never be more reliable than the data, and will often be less so. Work in which 4-figure logarithms have been used should usually be 'rounded off' to a 3-figure result because the fourth figure, though accurately obtained, is not reliable. Similarly, if the value $\frac{22}{7}$ is used for $\pi$, as a rule not more than two figures of the result will be reliable.

Some teachers find it difficult to appreciate this distinction between accuracy and reliability; the whole question of approximation comes to be regarded as an academic refinement, receiving more unsatisfactory treatment than any other aspect of arithmetic. It is highly desirable that the *teacher* should understand why the final figure in a table of logarithms (or trigonometrical ratios) is not reliable; a minority of children may also be brought to appreciate the basis and usefulness of different approximations to the value of $\pi$.

**6·851.** To understand the effect on the result of a computation when the data are approximate it is best to compute results in limiting cases. For example, suppose a rectangular area measures 86 ft. by 37 ft., to the nearest foot; the problem is to consider the possible error in taking the simple product 3182 sq. ft. as the area. If 86 ft. is correct to the nearest foot the length is somewhere between 85·5 ft. and 86·5 ft., these being the limiting values; similarly the limiting values for the breadth are 36·5 ft. and 37·5 ft. It follows that we can only say with certainty that the area lies between (85·5 × 36·5) sq. ft. and (86·5 × 37·5) sq. ft., i.e. between 3120·75 sq. ft. and 3243·75 sq. ft., a result which at first sight usually surprises the pupil. Examination of the result then brings out the fact that we can get 'agreement' between the two products only at the thousands figure: we cannot give with certainty even the nearest hundred, although in some examples this proves to be possible. The process of multiplication has thus reduced the number of reliable figures. Although the further development of such considerations may be beyond the powers of most modern school pupils, it should be possible for them to reach some appreciation of the reliability of figures; such appreciation would at least rule out the computation of results at unnecessary length which in the past has caused so much frustration and waste of time.

# CHAPTER 7

# OTHER TOPICS MAINLY MATHEMATICAL

## 7·0. Their Nature and Use.

There are a number of special applications of Mathematics, such as surveying, navigation, astronomy, etc., which can be of particular value in the Modern School. Their chief asset is that they act as a carrier or medium to introduce and to display the Mathematics we wish to teach. They show Mathematics as a tool subject, providing solutions to innumerable practical problems which arise. Their use may be restricted to a few lessons where they are used to show a particular mathematical idea or to give practice in some mathematical skill. For example, the apparent rotation of the stars around the Pole Star can be used to give interesting practice in angle measure, with no further reference to astronomy. On the other hand, many of these special fields can provide a course which may occupy one or two periods a week in the four years of the Modern School. It is this latter aspect which we are mainly concerned with here, although the teacher should not miss incidental opportunities to illustrate particular ideas or usages.

**7·01.** It must be firmly emphasised that these special fields are employed as a means of teaching Mathematics. As in so many mathematical skills, it is possible to teach the superficial techniques without the pupils becoming aware of the fundamental ideas involved. This should be avoided at all costs. The real purpose in teaching surveying or navigation in school is to develop further each pupil's mathematical knowledge and insight. That pupils also acquire the techniques of these subjects is of secondary importance: the teaching must clearly be directed so that they are guided to discover the fundamental mathematical ideas upon which the techniques of the special applications are based.

**7·02.** In the following sections, surveying and navigation are treated in some detail as examples of special applications. Briefer comments are made on drawing, mechanics and statistics.

## 7·1. Surveying.

*7·10. The aims of teaching surveying.*
*7·11. Measurement of lines.*
*7·12. Measurement of angles.*
*7·13. Indirect measurement of heights and distances.*
*7·14. Area surveys.*
*7·15. Traverse surveys.*

*7·16. Plotting ground levels.*
*7·17. Making a syllabus for surveying.*
*7·18. Apparatus.*

### 7·10. The Aims of Teaching Surveying.

Surveying provides an ideal medium through which a wide range of Mathematics may be taught. It is an excellent ‘ carrier ’ for geometry, and also offers many opportunities for the practice of arithmetical techniques. More advanced surveying work can make a realistic use of trigonometry. Surveying is invariably a popular subject with pupils since its practical nature makes a decided appeal, both to boys and girls.

There is another valuable aspect of surveying as a topic in schools. It is important not only mathematically, but for the wider education of the pupil, that he should be given training in ‘ attack ’ on the problems which may arise in various real circumstances. Surveying is an excellent source of such problems, ranging from very simple ones which the weakest of modern school children can tackle to problems demanding a sound mathematical background.

**7·101.** We must emphasise once more that in including surveying our main aim is to teach the fundamental Mathematics upon which surveying is based and to make use of the many opportunities it offers. This means that professional surveying methods are often discarded in favour of other methods which display to better advantage the Mathematics used. For example in 7·142 many ways are shown of setting out right angles. Of these, the least satisfactory from the mathematics teacher's point of view is the professional surveyor's cross.

On the other hand, all normal surveying conventions should be followed as closely as possible. Standard map symbols should be used, the surveyor's field-book method of recording adopted (7·146). These form the language of surveying, and are forms of expression which the pupils should acquire.

**7·102.** In the initial stages much simple work can be done near or around the classroom before embarking on more ambitious field work. There is considerable advantage to be gained in using apparatus of the simplest type since its purpose is then clearly and easily seen. The complexity of, say, a sextant or theodolite, hides its function from many pupils. Of particular value are the older navigational instruments: the Jacob's staff, astrolabe, etc.

Every occasion should be taken to link the surveying with other subjects and with various sections of Mathematics. Many such opportunities may arise, for example, in Geography (5·1).

There are certain geometrical ideas which a child acquires through everyday experience; for example the shortest distance between two points, and horizontal and vertical directions. The child will use these ideas without being fully aware of them. Geometrical knowledge of this nature can be used in outdoor work without classroom preparation, but the work should compel the pupil to become fully conscious of the implicit ideas; he should acquire the appropriate vocabulary at the same time.

There are many other geometrical properties, such as Pythagoras' theorem, and the augle in a semi-circle, which are not intuitive (6·58).

Before these are used in surveying, the pupil must become acquainted with them. Their practical use may then ensure that they are grasped and absorbed.

**7·103.** There are many difficulties which will confront the teacher of surveying, but most of them will be overcome by careful preparation and with increasing experience.

The main difficulties are:

(*a*) Organisation of a whole class with tasks suited to pupils' individual abilities.
(*b*) Organisation of apparatus and equipment, distribution, porterage, collection.
(*c*) Preparation of suitable simple apparatus.
(*d*) Arranging within the time-table working periods long enough to allow a piece of surveying work to be completed.

Only experience will show how much of the practical work can be initiated in the field and how much must follow discussion and preparation in the class-room beforehand. Opinion suggests that the value of what is achieved in the growth of mathematical ideas and the gain in experience and appreciation are well worth the time and effort expended, although the actual amount of recorded work may seem slight.

The following sections indicate the kinds of exercises which may be undertaken. The order of sections is not a teaching order but in each section the later methods shown are generally more difficult and should be taken later in the course. It will be seen that certain geometrical notions, such as those of horizontal and vertical directions and their relationship, permeate much of the work and will be encountered by the pupils in widely differing circumstances. It is the teacher's task to direct the pupil's attention to the essential Mathematics in these cases so that he appreciates the fundamental mathematical notion which is applied to meet a variety of situations.

### 7·11. Measurement of Lines.

Surveying makes much use of the distances between observed points, and it is clear that these distances must be measured along the straight line joining the two points. An early exercise in surveying must therefore be the setting out of a straight line. Incidentally this work helps towards an appreciation of the geometrical notion of a straight line since we are measuring a line of which only two points are visibly defined. For short distances we may give it concrete form by stretching a string from one point to the other, but the idea of the line and the possibility of measuring it persist even when the string is removed.

Standards of accuracy of measurement should be considered. When measuring a distance of 200 ft. or more in a field or along a road, measurement to the nearest inch is unnecessary. When the map is drawn the width of the pencil lines will probably represent 2 in. or more, according to scale. The difficulty of measuring a long distance might be demonstrated (6·821, 6·822). Groups can measure independently a fixed distance of about 200 ft. or more, each group giving their result to the nearest inch. The discrepancies will indicate the folly of stating such a measurement to an unwarranted degree of accuracy. For a distance of this order measurement to the nearest foot or link is generally accurate enough for most school surveying.

**7·111.** The following means of direct measurement of distances are generally available:

(*a*) *Chain.* The surveyor's chain is 66 ft. (1 chain) in length, and is divided into 100 links. This is one of the few instances in the British system of measures where a decimal reduction factor is used. Measurement in chains and links enables land areas in acres to be readily found (1 acre = 10 sq. chains), and provides a valuable source of real decimal calculations. A distance of, say, 2 chains 31 links is recorded as 2·31 chains.

The engineer's chain is 100 ft. long, divided into 100 links of 1 ft. It is rather heavy for school use.

Each tenth link on a chain carries a brass tablet (Fig. 38). The shape of the tablet indicates its distance from the nearer end of the chain.

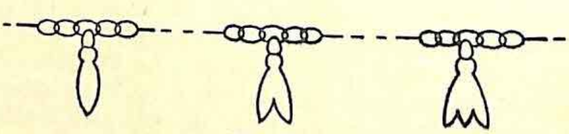

FIG. 38.

(*b*) *Tape.* Tapes can be obtained in a variety of lengths; the most useful are those 66 ft. or 100 ft. in length. 66 ft. tapes are often divided into links on the reverse side.

(*c*) *String, wire.* Few schools will possess a supply of chains or tapes adequate for all the demands of the surveying classes. 66 ft. lengths of string, light rope or wire, knotted or marked at 10-link intervals may be used in their place. Shorter distances can be estimated, or measured by a 10-link length marked at 1-link intervals. Wire is less convenient to handle than string, but is much more accurate since it does not stretch as readily as do string and rope.

(*d*) *Pacing.* A distance of 1 chain, 100 ft., or 100 yds. should be marked out and the pupils count their paces along it. The information so obtained can be used for setting out distances of 50 ft., 100 ft., etc. Each pupil may also find his average pace by calculation, and use this to measure distances by pacing.

For shorter distances the *heel-and-toe* method can be used. The units of measurement will be ' foot-lengths ' i.e. length of pupil's own foot, but these may be converted to more conventional measures.

(*e*) *Wheel, trundle or waywiser.* The number of rotations of a wheel of known circumference (hoop, bicycle wheel, etc.) may also be used as a measure of distance. The first measurements may remain as ' wheel-lengths ', but later measurements can be changed to feet, or links.

**7·112.** When the distance to be measured is longer than the chain or tape in use, it becomes necessary to ensure that the measurement is made along the straight line between the two points. One person, the *follower*, remains at the first point holding one end of the tape, while another, the *leader*, sets off in the direction of the second point, laying out the tape as he goes. When he reaches the end of the tape he is *lined up* by the follower looking along the line of sight of the two points. The leader makes a mark on the ground at the end of the tape, or inserts an *arrow* into the ground. An arrow can be a metal skewer with some bright material, red ribbon, tied on so that it is easily seen. The tape is then taken forward until the follower reaches the arrow. The whole process is then repeated, another arrow inserted, and so on until the distance is measured. As the follower goes forward each time, he takes the arrow from the ground. The number of arrows he has tells the number of times the full length of the tape has been laid out.

When using pacing or the rotations of a wheel it is necessary to devise some other means of keeping a straight path. A *ranging rod*, a brightly painted 6 ft. pole, is set up beyond the second point, and on the continuation of the line of sight of the two points. As the pupil paces he keeps the second point and the ranging rod always in line.

### 7·12. Measurement of Angles.

The measurement of angles is another basic surveying technique. Some class-room preparation is needed before using angle measure in outdoor work. The right angle is exceedingly common in everyday experience. The concept of angle as a measure of rotation or change of direction is important, and needs introduction (6·511, 6·512).

**7·121.** The following exercises use or give practice in angle measurement:

(*a*) If a pupil points N and then swings with his arm extended to point E, he rotates through a right angle. The intermediate points NW, SE, etc., involve estimating $\frac{1}{2}$ right angles. $\frac{1}{4}$ right angle can also be estimated.

Very rough mapping may be carried out by estimating approximate compass bearings of objects from a given point. Distances of the objects may be found by pacing.

(*b*) Angles between directions of objects can be fixed with the aid of a sighting rule. This may be a ruler carrying a pin or thin nail at each end of its axis. A sheet of paper is fastened to a table or horizontal board out-of-doors (7·18). The sighting rule is placed upon it and rotated until an object lies on its line of sight. A pencil line is then drawn along the edge of the rule. It is then rotated to sight another object, and another line is drawn, so fixing the angle between the directions of the two objects. It is advisable to let the sighting rule rotate about a pin through a hole in its axis, or at one end of a side of the rule.

From a point in the playground, each corner may be sighted and sighting lines drawn. If the distances of the corners are then found, a plan may be drawn. This is the simplest form of plane table work.

Another useful exercise is to sight outstanding objects from a point on high ground and estimate their distances. The results can be checked by reference to a 6 in. or 25 in. O.S. map of the area.

(*c*) A N–S line may be established through a *shadow stick*. A pole is set up in a vertical position, and from about 10 o'clock onwards on a sunny day the position of the tip of its shadow is marked at intervals of about ¼ hour. A North direction is given by the shortest shadow, but this is not easy to determine directly. The method shown in Fig. 39 is more accurate. One end of a length of string is fastened to the foot of the shadow stick. The other end swings to cut the shadow path at *A* and *B*. These give the positions of equal shadows before and after noon, and the shortest shadow lies midway between them. The distance *AB* can be measured and halved, or the arc *AB* bisected by a geometrical method as shown.

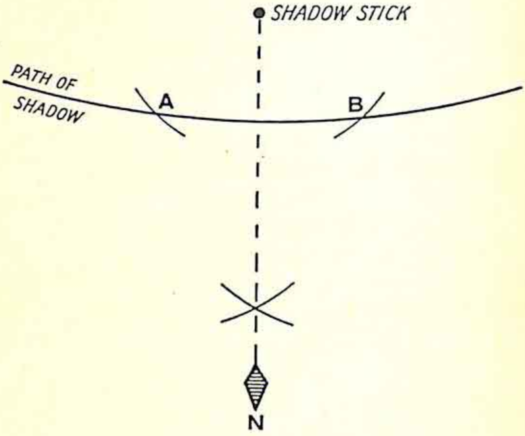

FIG. 39.

A N–S line should be marked in the playground or any other convenient place. It may be used as a reference for finding the compass bearings of various points. At first this can be done in terms of ½ or ¼ right angles, but more accurate measurement needs the division of the right angle into 90 degrees. One of the simplest ways of defining a direction is by relative bearings, i.e. the clockwise angle from some stated direction. Unfortunately, true N, compass N, and magnetic N are not identical, but their differences should be ignored until later stages in the work. Fig. 40 shows an anglemeter, a simple instrument for measuring horizontal angles.

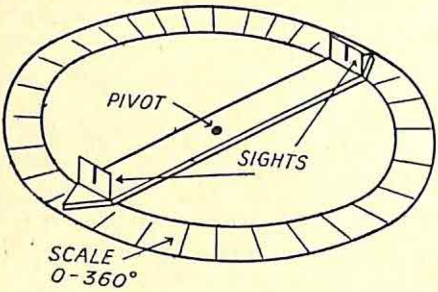

FIG. 40.

(*d*) A S–direction may also be found approximately by the 'watch and sun' method. A watch is held horizontally with the hour hand pointing to the sun. S lies halfway between the hour hand and 12 on the watch face. By this method, at 12 o'clock the hour hand will point S. The hour hand rotates through 360° in 12 hours, the sun appears to move through 360° in 24 hours, so the hour hand is moving twice as fast as the sun. Note that this method does not take into account local time, summer time, or the equation of time.

(*e*) The *shadow stick* can also introduce the measurement of angles in a vertical plane. The lengths of the stick and of its shadow are measured at some noted time. A scale drawing is made, preferably on squared paper, and from it the angle of elevation of the sun may be found. By taking a number of readings throughout a day, the variation of the sun's elevation may be recorded. A graph is a valuable means of displaying this information.

Another useful exercise is to take a series of readings at the same time of the day at weekly intervals.

A direct reading of the sun's elevation may be found by placing a large protractor at the shadow tip, and stretching a length of cotton from it to the top of the shadow stick.

(*f*) The slope of a road or hill, can be determined by measuring distances along the road and finding the rise in height. The latter may be found through a vertical pole marked in inches, and one of the horizontal sighting devices mentioned in 7·161. The statement of the slope as, say, 1 in 10 involves the notion of ratio; problems based on slopes make use of proportion (6·183 *c*, 6·243).

## 7·13. Indirect Measurement of Heights and Distance.

The first ideas of trigonometry may be infiltrated quite early in the surveying course through situations such as those described in 7·121 (*e*) and (*f*). The properties of similar figures have already been used in scale drawing and in the work on slopes. It is now advisable to show more clearly the mathematical notions we have been using.

A series of similar right-angled triangles (Fig. 41) are drawn on squared paper; and the sides containing the right angle are measured. The ratio of the length of the perpendicular to that of the base is easily established as ac onstant. In early work, it is better to choose some simple ratio for the first pair of sides, e.g. 3 : 5, rather than to fix the angles of the triangle. The ratio may be given some name such as *multiplier* or *operator*. It is the quantity by which one side must be multiplied in order to find the length of the other side.

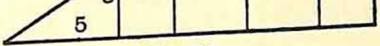

FIG. 41.

**7·131.** The finding of heights by shadow stick or set-square methods can then help to establish this first idea of the tangent of an angle (6·61). A rod of known height is set up vertically; its shadow is measured at intervals to find when the lengths of the rod and the shadow form a simple ratio, such as 1 : 1, 1 : 2, etc. Shadows of buildings, trees, etc., are also measured at the noted time. Their heights may then be found by a simple calculation.

**7·132.** The set-square method uses set-squares in which the lengths of the sides containing the right angle also form some simple ratio, 1 : 1, 1 : 2, etc. A pair of sights are mounted on the set-square so as to give a line of sight parallel to the hypotenuse; also a plumb-line to ensure that it is held correctly (Fig. 42). The observer moves away from the base of an object until the top of the object is seen along the line of sight. The distance from the base of the object is then measured. The height of the object can then be found by multiplying the base length by the appropriate multiplier for the set-square in use. In this and similar methods using a sighting device held at eye-level, the observer's eye-level height must be added to the calculated height.

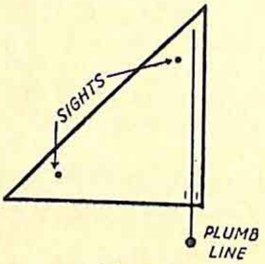

FIG. 42.

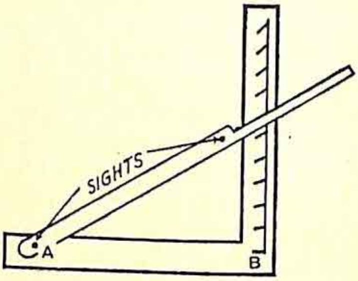

FIG. 43.

**7·133.** There are many other forms of simple height measurers which use the same basic principles. In Fig. 43 a pointer is pivoted at *A* and swings over a vertical scale 10 ins. long. The scale is divided into inches and tenths of inches; the inch divisions are marked 0, 0·1, 0·2, 0·3, etc., starting from the lower end. The distance *AB* is also 10 ins. The top of a building, or tree, etc., is sighted along a pair of sights carried on the pointer. The position of the pointer on the vertical scale gives the ratio of the lengths of the vertical and horizontal sides of the triangle; it is the factor by which the length of the base line must be multiplied to give the height.

Alternatively, a vertical rod of fixed height may slide along a horizontal bar which carries a scale (Fig. 44). This device is very similar to the Jacob's staff used in navigation a few hundred years ago.

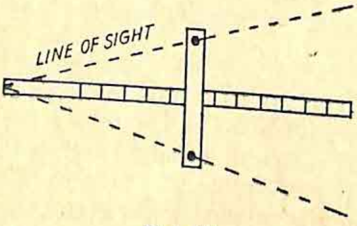

FIG. 44.

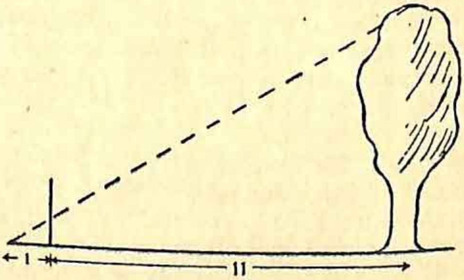

FIG.

The Scout's method of finding a height also uses similarity and is illustrated in Fig. 45. A vertical staff marked in inches is set up 11 paces from the foot of the object. The observer, at ground level, and one pace from the staff, observes the reading where the line of sight of the top of the object cuts the staff. Each inch on the staff corresponds to one foot in height of the object.

**7·134.** The height of an object may also be found by measuring the angle of elevation of the top of the object at some measured distance from the base with a *clinometer* (Fig. 46) or some other angle measuring device. Scale drawing can then be used, or the height may be calculated by an extension of the ratio methods already employed.

A series of right-angled triangles is drawn with base angles 5°, 10°, 15°, 20°, etc. In each case the ratio of the lengths of the perpendicular and base is found. A table of angles and their corresponding ratios is made; this is, of course, a simple form of the table of tangents (6·61). Intermediate values not included in the table may be found if a graph is drawn, plotting values of angles on the horizontal axis and ratios on the vertical axis. The height of an object may then be calculated if the length of the base line and the angle of elevation are known. From the graph the ratio of the perpendicular to the base for the measured angle of elevation is found. Simple proportion methods are then used to find the required heights.

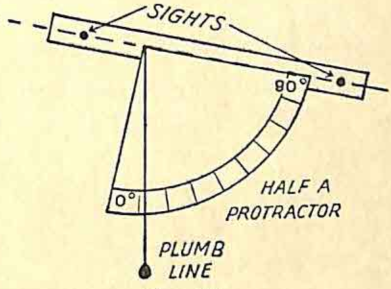

FIG. 46.

After the graph has been used in this way for some little time the table of tangents may be introduced.

**7·135.** Extensions of the above methods are required when the base of the object is inaccessible, so that it is not possible to measure the observer's distance from the foot of the object. One of the simplest methods in these circumstances is to take readings with the 1 : 1 and 1 : 2 set-squares (Fig. 47). The distance between *A* and *B* is equal to the height of the object. *A*, *B* and *O* must lie on the same straight line.

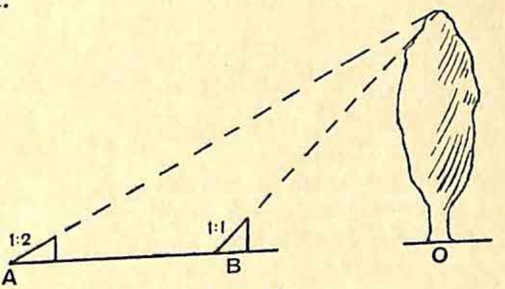

FIG. 47.

The same exercise may be carried out using a single set-square with its sides in the ratio 1·618 : 1. The nearer reading is taken with the larger side in the vertical position, the further reading with it in the horizontal position. The distance between the two points of observation is then approximately equal to the height of the object. The proof that the ratio

1·618 : 1 gives the desired result is beyond the ability of most modern school children, but the fact that it is the required ratio can be demonstrated by drawing methods, and it can be established numerically that 1·618 is the reciprocal of ·618.

The clinometer may also be used in a similar manner; two readings of the angle of elevation are taken at a measured distance apart. The height can then be found by scale drawing. Able pupils may be able to obtain a general solution in terms of *AB* and the tangents or cotangents of the angles (6·361, 6·63).

**7·136.** Further geometrical properties may be demonstrated when finding distances which cannot be directly measured, such as the width of a river or pond. For the sake of convenience ranging rods are generally set up a few feet from each river bank, and the distance between them is measured.

Each of the following methods uses some particular geometrical notion.

(*a*) A third ranging rod, *C*, is placed so as to form, with *A* and *B*, an equilateral triangle (Fig. 48). Some method of setting out an angle of 60° must be devised, e.g. a loop of string divided into three equal sections.

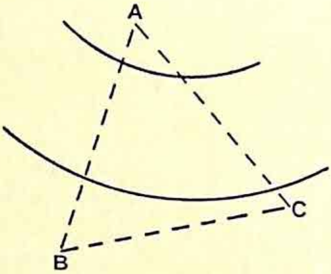

FIG. 48.

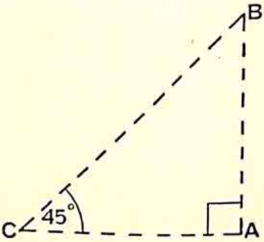

FIG. 49.

$$AB = BC$$

(*b*) A line perpendicular to *AB* is set out at *A*, see 7·142. A point *C* is found on this line so that $\angle ACB = 45°$ (Fig. 49). The triangle *ACB* is an isosceles right-angled triangle.

$$AB = AC$$

(*c*) A line *AC* is set out perpendicular to *AB* (Fig. 50). *D* is its midpoint. *CE* is set out perpendicular to *AC*. The observer moves along *CE* until *D* and *B* are on the same line of sight. The triangles *ADB* and *CDE* are congruent triangles.

$$AB = CE$$

(*d*) A third ranging rod *D* is set up on the line of sight *BA* and some distance from *A* (Fig. 51), *AC* is set out perpendicular to *AB*. $\angle ACB$

is measured; $\angle ACE$ is set out equal to $\angle ACB$. The triangles $ACB$, $ACE$ are congruent.

$$AB = AE$$

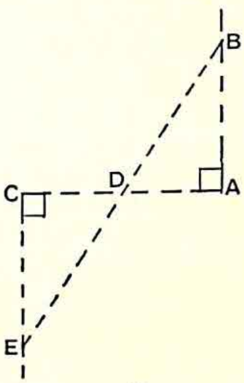

FIG. 50.

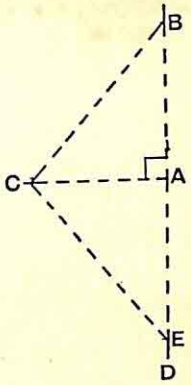

FIG. 51.

(*e*) In methods (*a*) to (*d*), the points $A$ and $B$ must be intervisible. If $A$ and $B$ cannot be seen from one another, a point $P$ is taken from which $A$ and $B$ can be seen (Fig. 52). $PA$ and $PB$ are measured and their midpoints, $C$ and $D$, found. $CD$ is measured.

$$AB = 2CD$$

(*f*) An alternative method to (*e*) is to produce $AP$ and $BP$ to $D$ and $C$ respectively, making $PC = BP$ and $PD = AP$ (Fig. 53). $CD$ is measured. The triangles $ABP$, $CDP$ are congruent.

$$AB = CD$$

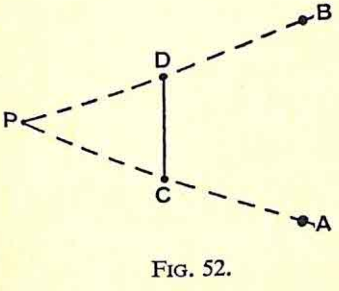

FIG. 52.

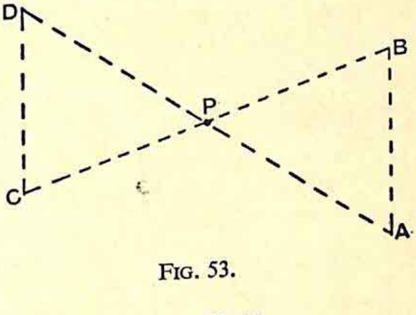

FIG. 53.

In both (*e*) and (*f*) $CD$ also gives the direction of $AB$. (See also 7·143.)

### 7·14. Area Surveys.

Some simple methods of surveying useful in making a map have already been mentioned (7·121 *a* to *c*). A considerable amount of more advanced surveying work is necessary in mapping areas, but elementary exercises can be introduced very early in the course.

**7·141.** Possibly the simplest exercise is the measuring of a rectangular site, assuming the angles to be right angles and drawing the plan on squared paper. Rectangular playgrounds, etc., are by no means common, but games pitches, tennis courts, or even a site pegged out on a field may be used.

The converse exercise, setting out a rectangular area such as a football or netball pitch, is undoubtedly of greater value. It demands not only the marking out of given distances, but also the setting out of perpendicular lines, parallel lines, etc. The shape should be tested after completion by measuring those lines and angles which have not been set out to required dimensions. For example, it is likely that a rectangle would be set out by marking out three of its sides and two angles. The other side and angles should be measured, also the diagonals.

**7·142.** Some of the many ways of setting out lines perpendicular to each other are given below:

(*a*) The surveyor's cross carries sights which give two lines of sight perpendicular to each other. Very simple versions may be made by using screw eyes for sights mounted on a T-square or set-square.

(*b*) A 12 ft. loop of string is knotted so as to give lengths of 3 ft., 4 ft., and 5 ft. The string is stretched so as to give a right-angled triangle with sides 3 ft., 4 ft., 5 ft. A 5, 12, 13 triangle is equally useful. This historic method, said to have been used by the Egyptians, could be a first introduction to Pythagoras' Theorem (6·587).

(*c*) Fig. 54 illustrates a method which can be used to drop a perpendicular from a point *A* to a given line. One end of a tape is fixed at *A*, and the tape stretched so as to cut the line at *B*. The midpoint *C*, of *AB* is found. The tape is held at *C* while the end *B* swings to cut the line again at *P*. $\angle APB$ is the angle in a semicircle, diameter *AB*. *AP* is, therefore, perpendicular to *BP* (6·588).

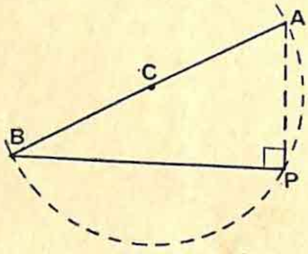

FIG. 54.

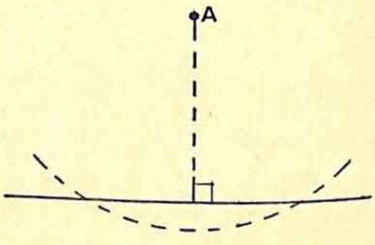

FIG. 55.

(*d*) One end of a tape is fixed at *A* (Fig. 55). The reading on the tape is noted as it swings across the base line. The position of the tape giving the minimum reading is the required perpendicular. (The shortest distance from a point to a line is the perpendicular to the line.)

(*e*) One end of a length of tape is fixed at $A$ (Fig. 56). The other end swings to cut the base line at $B$ and $C$. The midpoint $P$ of $BC$ is the foot of the required perpendicular; $ABC$ is an isosceles triangle.

(*f*) Equal distances, $AB$, $AC$ are marked on the base line (Fig. 57). The two ends of a long length of tape are held at $B$ and $C$. The midpoint, $D$, of the tape is taken out so as to stretch the tape. $DA$ is perpendicular to $BC$; $DBC$ is an isosceles triangle.

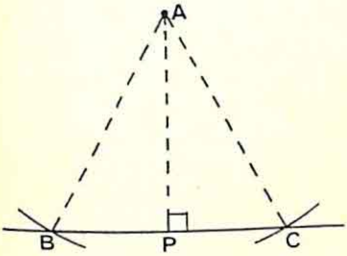

FIG. 56.

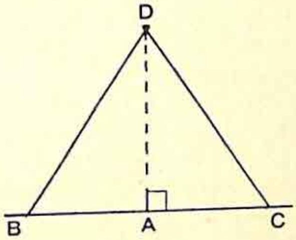

FIG. 57.

**7·143.** It is probable that the quadrilateral is the most common shape encountered when mapping areas. It is not, however, the simplest shape to measure. Few pupils realise that measuring the lengths of the four sides only does not fix the shape of the quadrilateral (6·584).

Most methods of mapping areas depend on dividing the area into triangles (6·532). It is advisable therefore to plot such triangular areas as are available. The most convenient method is undoubtedly that of measuring the three sides. Class-room work on the construction of triangles given the three sides follows naturally from such an exercise, and can proceed to other triangle constructions. The fixing of the position of a distant object can involve the use of the construction of a triangle given one side and two angles. A base line from which the landmark can be seen and which is as long as possible, is set out. The bearing of the landmark is measured from each end of the base line. This may be a compass bearing taken by a prismatic compass, or it can be the angle between the line of sight and the base line, and measured with an anglemeter (Fig. 40).

**7·144.** The position of a distant object can also be fixed by plane table methods (7·121*b*). A base line is measured as before. A sheet of paper is fastened to a *plane table*, i.e. a drawing board mounted on a tripod, or a light, portable table, which is placed over one end of the base line. A point, $A$, is marked on the paper to represent the position of the table. Sighting lines to the landmark and to the other end of the base line, $B$, are drawn. The length of the base line is marked off according to some convenient scale. It is advisable to estimate the distance of the landmark and choose a scale which will ensure that it can be plotted on the paper. The table is then taken to the other end of the base line. The sighting rule is placed along the drawn base line, $AB$, and the paper or table rotated until the directions of the drawn and real base lines coincide. A sighting line to the landmark is then drawn through the point B. The intersection of the two sighting lines from $A$ and $B$ fixes the position of the landmark.

It will be noticed that we are, in fact, determining the shape of a triangle by measuring a side and fixing two angles.

**7·145.** The methods of measuring triangles are later extended to the measurement of other areas by dividing them into triangles. A quadrilateral may be determined by measuring the four sides and a diagonal; other polygons are similarly measured. Alternatively, the plane table method of 7·144 can be used. The corners are fixed by sighting lines drawn from the two ends of a base line set up inside the area.

**7·146.** An alternative way of surveying an irregular area (Fig. 58) introduces the surveyor's fieldbook method of recording. *BE* and *FD*, known as *offsets*, are set out perpendicular to the diagonal *AC*. The distances *AE*, *EB*; *AF*, *FD*; and *AC* are measured. It will be seen that we are establishing the positions of *B* and *D* by a system of co-ordinates. The method of recording is shown in Fig. 59. Distances along the diagonal *AC*, generally known as the *survey line*, are entered in the double column in the centre of the record, which is read from the bottom upwards. Distances to the left or right of the survey line are entered on the left or right of the page, together with any necessary details. No attempt is made to provide a scale diagram, but sufficient information is shown to allow the record to be used for drawing the plan.

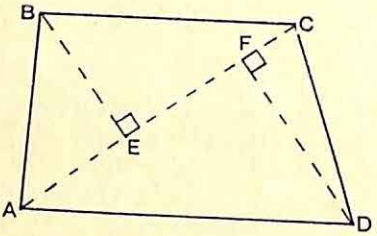

FIG. 58.

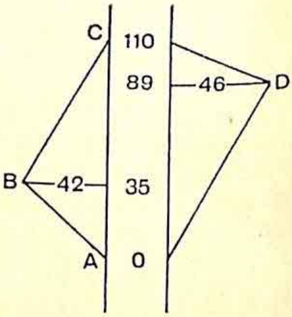

FIG. 59.

It will be noticed that the measurements made are the bases and perpendicular heights of triangles, so that areas can be quickly found.

The fieldbook system of recording is an interesting example of an unusual departure from common procedure in the way in which the recording is made, and later read, starting at the bottom of the page and working upwards. It is designed to suit particular circumstances and is by far the simplest method of recording to use in practice.

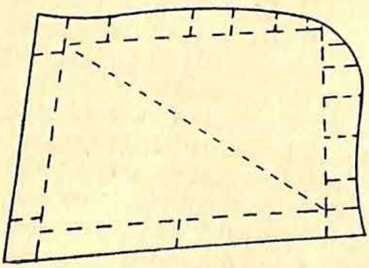

FIG. 60.

**7·147.** The use of offsets and the fieldbook method of recording becomes invaluable when we wish to map an area with irregular boundaries (Fig. 60). A framework of survey lines is set up within the area and offsets from these to the boundary are measured. The offsets are not placed at regular intervals but at sufficient points to allow the boundary to be determined.

### 7·15. Traverse Surveys.

It is sometimes necessary to map a narrow stretch of country, e.g. for a new road or railway, and this may be carried out by means of a *traverse survey*.

**7·151.** It is unlikely that one survey line only will suffice, and the directions of the survey lines used must be fixed. This can be done in a number of ways. One simple method is to measure the bearing of each survey line with a bearing compass. Thus before starting to measure distances along *AB* (Fig. 61), the direction of *AB* relative to North is found. Details of the areas on either side of the survey line are then fixed by offsets and recorded by the fieldbook method. On arriving at *B* a *backbearing* is taken, i.e. the direction of *A* from *B* is found. This should, of course, differ by 180° from the forward bearing of *B* from *A*. The following survey lines, *BC*, *CD*, etc., are treated in the same manner.

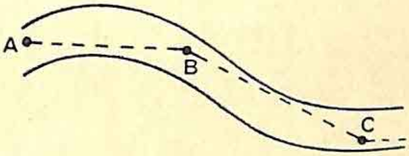

FIG. 61.

**7·152.** Another method of fixing the directions of the survey lines is to measure the angles between each pair of survey lines with an anglemeter or some other device for measuring horizontal angles. The angles may also be fixed by drawing sighting lines on paper. It will be realised that these methods only determine the relative directions of the survey lines; their directions with regard to North are not known. Details can be filled in by the method of offsets.

**7·153.** Plane tabling may also be used to fill in the details of a traverse survey. Each survey line is used as a base line, and sighting lines to all objects, etc., are drawn from both ends. Thus, from *A* (Fig. 61) a sighting line to *B* is drawn. *AB* is measured and drawn to a convenient scale. Sighting lines to all objects from *A* are drawn. The table is then taken to *B* where it is set up and orientated so that the drawn line *AB* and the survey line *AB* coincide in direction. Sighting lines are then drawn to the objects around both the survey lines *AB* and *BC*. This process is continued until the traverse is completed.

### 7·16. Plotting Ground Levels.

The surveying work described so far takes no account of the varying levels of the ground. The measuring or comparing of ground levels is known as *levelling*. It is not easy to measure the exact height of any point above sea level, although benchmarks are to be found in all parts and the heights of these are given on the large scale Ordnance Survey maps. It is possible, however, to measure the relative heights of a series of points.

If work is started from a benchmark actual heights above sea level may be established. For most school levelling, however, work is started from any suitable point, which is labelled 0, and heights are found relative to this starting point. Exercises of this character can form an excellent introduction to the use of directed numbers.

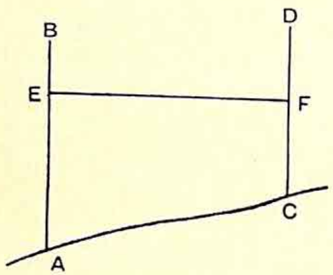

Fig. 62.

**7·161.** One of the simplest levelling exercises is to plot the profile of a sloping path or road. Fig. 62 shows one method of measuring differences of ground heights. A rod *EF*, 6–10 ft. long is held between two other vertical rods *AB*, *CD*, which are marked in inches. A spirit level placed on *EF* checks that it is being held in a horizontal position. The difference between the readings *AE*, *CF*, gives the rise in ground level from *A* to *C*. Readings are taken in a series of steps, each equal to *EF* in length.

A string may be substituted for the rod *EF*, and the steps can then be 20 ft. or more in length. A spirit-level suspended in a wire cradle from the centre of the string checks that it is being held in a horizontal position. It is essential, of course, that the cradle is placed at the exact centre of the string, otherwise the sag of the string will affect the reading.

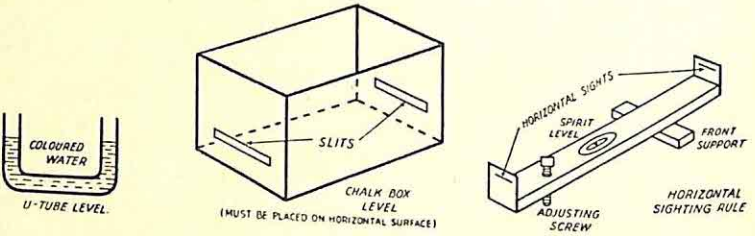

Fig. 63.

An alternative method is to use some device which gives a horizontal line of sight (Fig. 63). One of these is set up at *A* (Fig. 64) and readings of a levelling staff, marked in inches, are taken at successive positions, *B*, *C*, *D*, etc. The change in readings denotes the change in ground level.

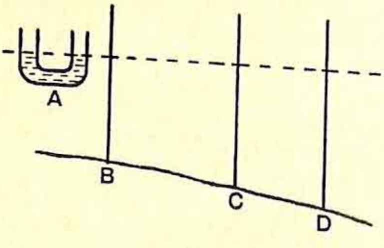

Fig. 64.

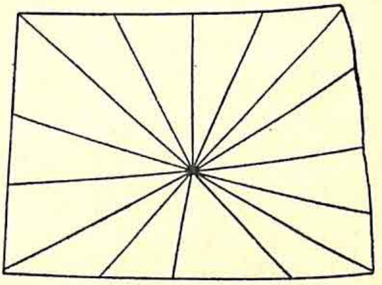

Fig. 65.

The mapping of a winding, sloping path forms an excellent combined exercise in levelling and traverse surveying.

**7·162.** An area may be levelled by a set of radial lines from some convenient central point (Fig. 65) or by a rectangular grid of lines (Fig. 66). In the latter case, if $A$ is taken as the starting point, the line $AF$ is levelled first to establish the heights of $B$, $C$, $D$, etc., relative to $A$. These points are then taken as the starting points of the lines $AG$, $BH$, $CJ$, etc.

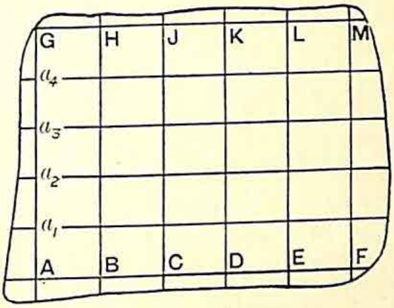

FIG. 66.

**7·163.** Most methods of finding heights over an area establish spot heights, i.e. the heights of a series of points along each levelling line are found. By far the most convenient method of showing ground heights on a map is by means of contour lines joining points of equal heights. To find these in the field is a slow and tedious procedure, and it is generally quicker to fix them by drawing the profile of each levelling line (Fig. 67). The positions on the levelling lines of points, say, 6 in. above $A$ are found from these profiles by interpolation. The points are then marked on the levelling lines on the drawn plan. A line drawn through them is the 6 in. contour line.

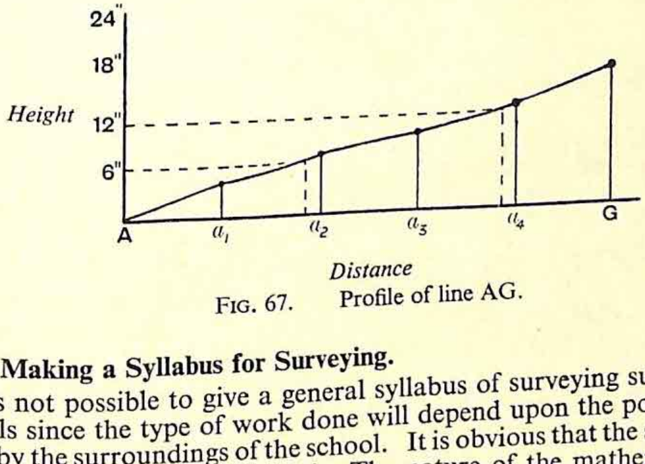

FIG. 67. Profile of line AG.

## 7·17. Making a Syllabus for Surveying.

It is not possible to give a general syllabus of surveying suitable for all schools since the type of work done will depend upon the possibilities offered by the surroundings of the school. It is obvious that the syllabus must be compiled with these in mind. The nature of the mathematical ideas used in the surveying must also be carefully considered; it is advisable to state these clearly in the syllabus so that the latter does not become a collection of surveying techniques. The syllabus should be regarded, in fact, as a statement which shows the ways whereby surveying can act as a guide to mathematical ideas. The column method of setting out a syllabus (5·32) is of value in this connection, the columns showing the surveying techniques side by side with the Mathematics involved in each.

**7·171.** For example, the notion that the position of a plane is determined by any two straight lines on it may be seen when testing a surface such as a plane table, with a bubble level. At first the level is likely to be used for testing in one direction only, but it will soon be realised that this test does not guarantee a horizontal surface. The conventional technique is to test in two directions at right angles to each other. It is not essential that the directions should be mutually perpendicular, although this probably provides the most sensitive test. If the surface is horizontal in any two directions, it must be a horizontal surface.

The step from the technique acquired incidentally in surveying to a grasp of the geometry involved will certainly have to be guided by the teacher. His task is, however, made much easier by the understanding acquired in the practical situation. If the step is not taken, then a valuable opportunity of adding to the pupil's fundamental mathematical knowledge has been lost.

### 7·18. Apparatus.

One of the several problems facing the teacher of surveying is the provision of an adequate supply of apparatus. The price of professional surveying apparatus places it beyond the means of most schools. Moreover, as already suggested in 7·102, the complexity of some professional apparatus obscures its mode of operation from most modern school pupils, so that they use it without understanding. Many schools make a feature of designing and constructing their own apparatus. Variety can be added to the surveying course by devising apparatus to suit particular tasks. Much of this can be of an extremely simple nature.

For example, an instrument for measuring horizontal angles can be made from an ordinary school ruler and a sheet of polar graph paper (obtainable from shops supplying drawing office equipment). The graph paper is marked in degrees, 0°–360°, and fastened to a table or other horizontal surface. A pin is placed at the centre of the graph paper, and a ruler held with its base against the pin. The observer looks along the edge of the ruler to sight the required object. Better sighting is obtained by fastening a milk straw to the edge of the ruler with cellotape, and looking through the tube formed by the straw.

Even with apparatus of such simple character as this a wide range of useful surveying work can be carried out.

## 7·2. Navigation.

*7·21. Preliminary work.*
*7·22. Development from elementary principles.*
*7·23. Symbols used in air navigation.*
*7·24. The air plot.*
*7·25. The triangle of velocities.*
*7·26. Position lines.*

**7·20.** In curiously varied ways, many of our pupils acquire plenty of skill in finding their way from place to place by land, and in so doing they frequently exercise judgment in matters of direction, speed, distance, map-reading and position-location generally. Some of the factors which influence their judgment may well have been explored during school lessons —for instance, in consideration of angle measurement, of compass points, relative bearings, scales, distances on maps and so on—or they may have been acquired, as are so many ideas, by incidental, individual means.

If, however, these pupils are to investigate the basic methods of navigation by sea or by air (and by this is meant the art of determining the position of a ship or aircraft, and of guiding it from one position to another) it will be necessary for them to clarify and consolidate all these concepts, and to add others to them. As with so many other studies, navigation can be followed at more than one academic level. In recent years it has been found possible and attractive to numbers of modern school pupils, and, as a rather more systematised study, it is becoming increasingly popular as a subject at ordinary level in the General Certificate of Education examination.

### 7·21. Preliminary Work.

Preliminary work will be similar whether navigation by sea or by air is the subject of more advanced study.

**7·211. Direction.** Firstly, preliminary work must ensure familiarity with angles, angle measurement, and ideas of direction and bearing, as well as of distance and simple ways of fixing position. Apart from varied exercises within the classroom, the active experiences of determining South from the sun's position at noon (7·121*c*), and, if possible, North from the North Star, are of great value in giving a pupil a secure feeling of carrying with him a direction-finding framework of reference which he can practise at will. Simple exercises devised within the school grounds may enable him to test this skill, combined, if desired, with others of a mathematical nature e.g. *From the school gate, locate a solid object* 20 *yards away in a direction* 050°. *Measure it and draw a plan and elevation of it to a scale* 1″ *to* 1′. Similarly, from a study of a large-scale local map, he might not only practise the identification of landmarks visible from school vantage points, but also, perhaps as an out-of-school activity, identify places from sketched extracts, given the direction and approximate distance from the school.

**7·212. Map reading.** Alongside this, familiarity should be developed with all essential features of topographical maps. The use of map grid or graticule, as well as of latitude and longitude in fixing position, may be practised together with general map-reading, using local Ordnance Survey maps. One common form of doing this is by means of a ' navigational ramble ', which may be organised and recorded in more than one way. It might well include a walking route, partly indicated by roads, partly by named landmarks, and partly by map reference or latitude and longitude. A simple form of ' navigator's log ' might well be kept for such a walk; in this, distances, times, directions, alterations of course, and, possibly, estimated times of arrival or general observations would find a place.

**7·213. The earth as a sphere.** If informed by such a background, pupils are the better able to grapple with the simple geometry of the sphere in its application to the earth. The method by which Eratosthenes first determined the radius of the earth is by no means beyond the understanding of able modern school pupils; the basic method of the Bedford Level experiment is likewise simple.*

**7·214. The globe and map projections.** It is well worth while considering great and small circles, meridians, latitude and longitude on a globe, and making comparisons between what is found there and what appears on a map of Mercator's projection. A brief and simple study of map projections can bring out many points of interest. The sphere has a non-developable surface, i.e. its surface cannot be formed from a plane as can, say, a rectangular box from a sheet of card. Hence all flat maps of the earth are distorted in some way or another. The basic methods of map projection may be demonstrated by letting a spherical glass flask represent the earth. Lines and shapes painted on its surface can be 'projected' by a small source of light (e.g. car headlamp bulb) placed in the centre of the flask on to a flat sheet of paper held touching the sphere. The distortion of shape and size in the flat projection is readily seen. Other forms of projection are given by projecting on to:

a cylinder of paper held so as to touch a great circle, generally the equator;
a cone of paper touching a circle of latitude.

Most map projections are refinements of these three basic forms.

The Mercator chart, commonly used in navigation, is a form of cylindrical projection in which distances and areas are exaggerated as we approach the Poles. The advantage of the Mercator chart for navigation is that all *meridians* are drawn parallel, although they are in fact curved lines on a globe. A straight line drawn from one point to another on such a chart will cut all meridians at the same angle; it is known as a *rhumb-line*. It is clearly more simple to travel along such a line since the direction is kept constant, and this is generally the path flown by aircraft for short distances. The fact that this is not the shortest distance between the two points on the surface of the earth (unless they lie on the equator or on the same meridian) is an indication of the distortion produced by the flat representation of a curved surface.

Further study of Mercator's chart will reveal its inequality of scale. Though the mathematical element in the distortion may not always be suitable for probing, practical measurements on the chart should establish that the scale of distance is variable, and that, to minimise errors, one should measure distance on this chart at the scale of the average latitude between the two places. Normally scales appear at the side of the chart both in *nautical* and *statute miles*. The value of having one unit linked with that of another measure may be a new conception to some pupils, so that, if only for this reason the equivalence of one nautical mile to that of 1 minute of latitude (1′), is worth introducing. In fact, airspeed indicators on aircraft are normally calibrated in *knots* (1 knot is the speed equivalent to 1′ per hour), so that measurements in nautical miles are to be recommended. It may not be desirable except as practice in the use of logarithms to devote much energy to conversions from one unit to another.

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* Both these experiments are described in *First Steps in Astronomy*, Ch. I, by P. F. Burns, published by Ginn.

**7·215. World times.** Some consideration may also profitably be given to *world times* and how they arise. The method by which the longitude of a position is determined by comparing local time, as found by sun or star observation, with Greenwich time is still a fundamental method in navigation.

### 7·22. Development from Elementary Principles.

Ideally, principles of navigation would be met and mastered in connection with sailing, boating or flying, and many boys, in a part-time capacity as sea or air cadets, have been able to follow this pattern. But interest in man's activities by sea or air is widely spread, and taken as a form of imagined experience this study can readily be given the great attraction of realistic sea or air procedures. For example, the charts of the plotting series used by the Forces, e.g. G.S.G.S. 4080 Mercator's Projection, Section English Channel N.W. 48/4, may be purchased from several agents,* and these, together with the introduction of plotting instruments used also by the Forces, lend a valuable air of verisimilitude to the work.

There are several thousands of men teaching in our schools today who had first-hand experience of navigation by sea or as members of air crews. They would find it relatively easy to adapt the material of their own basic training into a form suitable for their pupils. Others who are interested to venture into this work might wish to consult official and other publications on the subject.†

**7·221. Air navigation.** In a limited space it is possible to do no more than to outline one example of a form of development, known to be suitable to sound but average pupils. In selecting air navigation because of its general appeal, it is noted that the range of work discussed can be treated almost purely as an application of Mathematics, though excursions into related fields add richly to the interest.

It may be recalled that air navigation consists in charting an aircraft's position, and in using that information to guide it to another. The main elements in achieving this include the reading of topographical maps, the checking by position-lines, and the practice of so-called *dead reckoning* (D.R.) Navigation. Some attention has been given to map-reading, and it will need no emphasis that an accurate ground recognition, a pin-point, giving a *fix*, is the most valuable and certain way of determining position. It is, however, in deducing an aircraft's position after a definite period of time, for example, after reckoning what its track and ground speed are, that new mathematical work can best be introduced.

### 7·23. Symbols used in Air Navigation.

At this stage it may be useful to collect together a few of the more essential symbols and abbreviations in general use. Not all are required or need be introduced at once.

| | |
|---|---|
| A/c | Aircraft |
| S/C | Set Course |

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* E.g. E. Stanford Ltd., 12–14, Long Acre, London W.C.2.; S. D. Potter, 145, Minories, London, E.C.3.

† *Air Ministry Manual of Air Navigation*, Vol. I. H.M.S.O.; Crossley, F. F., *A First Course in Navigation*, Macmillan (written for A.T.C.): Kaspar, *Teach Yourself Air Navigation*, E.U.P.; J. K. Summers, *Practical Air Navigation Simply Explained*, Pitman.

| | |
|---|---|
| A/C | Alter Course |
| Co (T) | True Course |
| T.A.S. | True Air Speed |
| G/S | Ground Speed |
| T.M.G. | Track Made Good |
| K | Knot |
| E.T.A. | Estimated Time of Arrival |

| | |
|---|---|
| Course (Co) | Heading or direction steered by aircrafts. |
| Track (Tr) | Direction in which the aircraft travels over the ground. |
| Drift | Angle between Course and Track. |
| Port | To left. |
| Starboard | To right. |
| W/V | Wind Velocity. This is stated by direction, W/D, and speed, W/S; e.g. 210°/25K shows a wind of 25K *from* 210°. |

Air Position: shown by △ and time
D.R. Position: shown by □ and time
Fix: shown by ○ and time

### 7·24. The Air Plot.

The equipment needed by each pupil consists of a plotting chart, a good protractor (e.g. a Douglas or a complete circular type showing angles from 000° to 360°), dividers, a straight edge and a sharp pencil.

First of all, it is helpful to practise following on the chart the path an aircraft would take if there were no wind affecting its flight.

This Air Plot makes use of the *vector:* thus a line is drawn representing the direction and distance covered during the timed flight of the aircraft. With a little experience, simple exercises of the following sort can then be plotted. This and later examples, may be plotted on sheet NW 51/3[†]; scale 1 : 500,000.

0900 hours. GRIMSBY S/C 170° (T): T.A.S. 120K.
0920 hours. LYNN WELL LIGHT[*] (Fix).

Find (i) The Air Position at 0920 hours.
(ii) The wind effect for the 20 minutes.
(iii) The W/V.

This is tackled by marking a point at GRIMSBY, noting also the time 0900 hours, and from this drawing the vector to represent the aircraft's path *through the air*, i.e. in a direction 170° and for 40′, i.e. in 20 mins. at 120K. The 40 nautical miles are stepped off from 40′ marked on the latitude scale at the average latitude of the flight, (7·214). At its end, the Air Position at 0920 hours is marked by a △. Now, in fact, there has been air movement too, and the aircraft has moved *with* the air as well as through it. In consequence, the Fix at 0920 hours is, as given, at a different point from the Air Position. This is marked by a circle as shown in 7·23 Fix. The argument now applied is that the effect of the wind during the 20 minutes has taken the aircraft from the Air Position to the Fix. Hence the two are joined, and the actual distance and direction between them measured. A movement of 12 n. miles in 20 minutes implies a wind speed of 36K, and a movement in the direction of 060° suggests a wind direction of 240°.

### 7·25. The Triangle of Velocities.

Though in practice the technique of the Air Plot is sometimes used, a general mastery of the vector triangle may be considered a more valuable mathematical achievement. It calls for plenty of class discussion, from which it must clearly emerge that the actual path of an aircraft, as of a ship, is the result of the combined effects of two separate velocities. For the aircraft these are:

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[†] Obtainable from e.g., E. Stanford Ltd., 12–14, Long Acre, London W.C. 2.; S. D. Potter, 145, Minories, London, E.C. 3.

[*] Asterisk is a conventional symbol attached to 'Light'.

1. The speed *through* the air (T.A.S.) found by suitably correcting the reading on the pilot's Air Speed Indicator. This speed is in the direction of the Course, as set on the direction compass.
2. The movement *with* the air i.e. W/V, made up of W/D and W/S. These cannot be directly measured from an aircraft, but can be computed e.g. after making a suitable fix. Otherwise serviceable estimates may be obtained from the meteorological office.

As a result of these combined movements, the aircraft moves across the ground below it in a line called the Track, or Track Made Good, at the ground speed.

It has been noted that some of these six values, three directions and three speeds, can be read from or set on instruments; sometimes they can be calculated. Frequently it is possible to find those unknown by drawing the *triangle of velocities* which links them all together. The simple case illustrated in Fig. 68 uses the markings accepted for course, track and wind vectors.

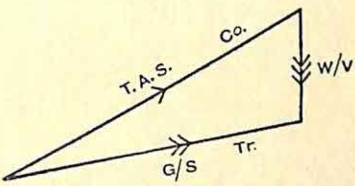

FIG. 68.

The three vectors forming the triangle are drawn to scale and in the correct directions. Two are *components*.

viz: (i) Course, at T.A.S.
(ii) Wind Velocity (at W/S from W/D).

These give the *resultant*, i.e. the Track direction flown by the aircraft at its G/S.

Pupils find it helpful to note that the two components will always follow each other in directional order.

A triangle of velocities may be drawn to scale and direction on any type of paper and to any suitable scale. For convenience solely, it is customary in the air to plot it straight on to the Mercator chart. The beginner often takes some time to sort out these facts, and to recall that the vector triangle does *not* show actual distances, but does need a uniform scale. Again, for this purpose though any suitable scale will do, the minutes of longitude (but not of latitude) marked on the chart are admirably suited for the plot.

**7·251.** A wide range of exercises is possible with the vector triangle, and the experienced teacher will find no difficulty in devising examples to meet different needs, and calling for more than one type of construction. Two typical cases are instanced:

1. *To find W/V knowing Co and T.A.S., Tr. + G/S.*
   1400 hours. A/C over Nore light* steering 040° (T). T.A.S. 120K.
   1415 hours. Felixstowe (Fix).

   Find (i) T.M.G. (ii) Distance flown (iii) G/S (iv) Drift (v) W/V experienced.

To do this, mark off a line from Nore* in the direction 040°, for the Course and join Nore* to Felixstowe for the T.M.G. Measure the actual distance covered along the Track in the 15 minutes (34′). This gives the G/S as 136K. The Track's direction is 033°, so that the Drift is 7° (Port).

Now draw the vector triangle, using the actual lines drawn for Course and Track, but applying a uniform scale to mark off T.A.S. along Course and G/S along Track. Complete the Triangle and measure the W/S on the same uniform scale and its direction by protractor.

2. *To find Co. to set, and expected G/S, knowing the required Track, the T.A.S. you will fly, and using the Met. forecast W/V.*

1500 hours. Cranfield (52° 03′ N 00° 38′ W) S/C Reading.
Forecast W/V 090°/24K. T.A.S. 110K.
Find (i) Track (ii) Distance (iii) Co. to set (iv) G/S expected (v) E.T.A.
1516 hours Spot Height (51° 39′ N, 00° 51′ W).
Find (vi) T.M.G. (vii) actual G/S (viii) actual W/V (ix) D.R. Position at 1520 hours.

### 7·26. Position Lines.

It was observed earlier that the navigator checks whenever possible by reference to a position line, i.e. a line along which the position of the aircraft is known to lie, although it is not precisely fixed. Among valuable types of position-lines are included:

(*a*) Ground features such as a railway line, canal, and desert pipe-line.

(*b*) Visual bearings; e.g. it is observed by compass that a town is due W. Hence, a line plotted due E from the town provides a position-line for the aircraft.

(*c*) Direction-finding wireless (D/F) bearings; e.g. if the bearing of a station from the aircraft is S.E. (135°), the aircraft must lie on a line drawn N.W. (315°) from the station.

Not only may use of these add variety, but they can be applied in different ways to give geometrical content to plotting exercises. In practice, this is more readily achieved in sea navigation, in which slower speeds permit more observations to be taken.

With adequate preparation, particularly interesting *plots* can be made by pupils working in pairs, plotting on charts or working simultaneously from allied topographical maps. On these, pin-point fixes can be made from a running commentary by the teacher, descriptive of the terrain. This calls however for a degree of skill and speed that should not normally be sought. If indeed it were possible, the more exacting mathematical opportunities, so far by-passed, might well be substituted.

### 7·3. Representational Drawing.

The fact that expression can take a variety of forms has been stressed in this report. Drawing is one of these forms which, by its nature, is particularly well suited to modern school pupils. Its value has been realised in many boys' schools; the fact that some of the pictorial projections are equally valuable to girls has not yet been given full recognition. Some of the values of a good drawing course are obvious: the use of geometrical shapes, the practice given in mensuration, scale, geometrical constructions, and the use of instruments, etc. A less obvious, but equally important, aspect is the training given in solid visualisation. Most people find difficulty in thinking in three dimensions; their mental images are flat ones, lacking in depth. Most real problems, however, occur with solid shapes, and the ability to think in solid form notably improves the attack on such problems. If this ability is to be trained through drawing, then it is clear that the drawings must be made from solid objects. Copying drawings from books or from the blackboard cannot possibly give the same enriching experience.

It may be pertinent at this point to state that the drawing course should not be the sole medium through which modern school geometry is taught. Its value is undoubted but it operates in a restricted field, and much worthwhile geometry lies outside its scope.

### 7·31. Drawing as a Part of Mathematics.

Drawing in schools is approached from two points of view: that of the crafts teacher who uses it as part of his technique, and that of the mathematics teacher who sees in the subject many valuable mathematical opportunities. If drawing is to be taken as a subject, it is the mathematical aspect which should be emphasised. Moreover, the crafts need only a restricted range of drawing; where the drawing is not too closely linked with the craft subjects, a much wider scope is possible and other forms, such as architectural drawing, may be included in the course.

The ability to read a drawing is valuable and this aspect of the work must not be neglected.

### 7·32. Conventional Practice.

The British Standards Office has drawn up certain standard practices to be observed in drawings, and these should be followed. A pupil who is not acquainted with conventional practice will obviously be at a disadvantage when reading drawings which have been made according to these standards.

### 7·33. Forming a Syllabus.

It is not intended to give here a description of the various forms of projection, etc.; for these any standard book on drawing may be consulted.

The following list gives the forms of drawing which should be considered when compiling a syllabus.

**7·331. Pictorial projection.** Most drawing courses in boys' schools start off with orthographic projection because this is needed in the craft workshops. The pictorial projections are simpler, however, and probably offer better starting points. Girls do not as a rule study the same crafts as boys, and will not therefore need such an intensive course in orthographic projection. They should however learn to make and to read plans and elevations, which are mathematically based on this form of projection but are often treated separately instead of as parts of a single drawing.

The following types of pictorial projection may be taught:

(*a*) Oblique.

(*b*) Isometric. (For school purposes isometric scale can be ignored. It is required only in some drawing examinations, and by a few firms.)

(*c*) Axonometric. This form, in which the plan retains its true shape is being increasingly used, especially by builders and architects. It is, to a certain extent, replacing geometric perspective on account of its much greater simplicity.

In all the above forms, rough sketching should be practised. Quarter-inch squared paper is useful for this purpose; it is possible to obtain paper with isometric ruling.

It may be necessary to point out that, while (*a*), (*b*) and (*c*) present pictures of objects, they do not give pictures *as seen by the eye*. They are pictures drawn according to certain conventions which ease considerably the drawing problems.

(*d*) Geometric Perspective. The three forms of pictorial projection already mentioned are simple; each could be taught by itself, or be the starting point of a drawing course. Perspective is much more difficult and can only be tackled by pupils with real drawing ability. It is, however, an excellent form of drawing and might well form the final stages of a good drawing course for able pupils.

**7·332. Orthographic projection.** This is the commonest form of working drawing, and is undoubtedly the best for showing constructional details, etc. The reading of such drawings does require training and experience, whereas the pictorial forms make little demand. This is the reason for the much increased use of the pictorial forms in factories which find difficulty in obtaining skilled labour.

In Britain first-angle projection is commonly used. Third-angle projection is standard in U.S.A. and is being increasingly used here. The Standards Office now accepts both forms, and it is possible that first-angle projection will be displaced at some time in the future.*

It is important that the teacher (and the really able pupil) should realise that both forms exist and, above all, that drawings should be consistent. In no case should a combination of the two forms be allowed.

**7·333. Sections.** Drawing the true shapes of sections makes a real demand, both on the pupil's previous drawing experience, and on his ability to visualise in three dimensions. It forms an essential component of any good drawing course. The same fundamental techniques are used whether the sections are those of a cone or pyramid, of a building or a staircase, or of a stretch of mountainous country.

**7·334. Nets of developments.** The unfolding of a cardboard box to show the plane shape from which it is made can be the start of a study of nets, or developments. Not all 3-dimensional shapes can be formed in this way: the sphere has a non-developable surface. The making of solid shapes from heavy cartridge paper or thin card is a fascinating pastime which most pupils enjoy (6·574).

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* See for example *Engineering Drawing Office Practice* (British Standards Institution), page 14 for further details.

**7·335. Enlargement and reduction.** The enlargement or reduction of drawings is not often considered as part of a technical drawing course, but the work illustrates some interesting mathematical notions. The drawing and its enlarged, or reduced, copy are similar figures; the use of a centre of similitude (Fig. 69) provides a valuable method in many cases. Fig. 69 can be envisaged as a 2-dimensional enlargement or as a 3-dimensional one by viewing the figures as pyramids cut by planes parallel to their bases.

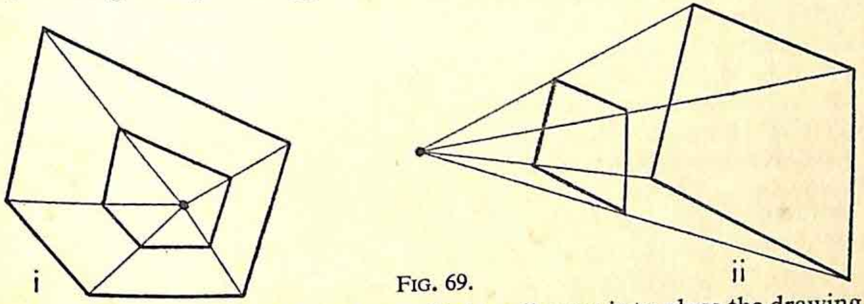


FIG. 69.

Another method which can be applied to all cases is to place the drawing within a rectangular grid. It is then copied using an enlarged grid (6·54). This method illustrates how the area is quadrupled when linear dimensions are doubled; it is particularly suitable for maps.

The standard instrument for enlarging, the pantograph, is based on similarity and certain properties of parallelograms. An experimental model can easily be constructed from Meccano strips.

**7·336. Curves.** A school drawing course may sometimes include the construction by geometrical means of composite curves, as, for example, sections of mouldings or the outlines of certain arches. One effective method emphasises the nature of a tangent to a circle; the curve is formed from arcs of circles drawn so that at each meeting point the two arcs possess a common tangent.

An elementary study of standard curves other than the circle, e.g. the ellipse, parabola, etc., and their methods of construction should also be considered (Appendix 1).

## 7·4. Mechanics.

The very early stages of the study of mechanics appropriate to courses of Mathematics or Science in Modern Schools are concerned mainly with observation, experiment, and description. The supplementing and underlining of everyday experience by experiment is not too difficult a task for the mathematics classroom. 'What happens?', 'How does it work?' are the questions asked rather than what are the principles underlying the happening, though some principles may emerge in the process of description. Most of the generalisations that do arise are mathematical in form so that whichever department, Mathematics or Science, treats the subject, co-operation between them is essential for healthy progress.

These principles may not yet be seen in complete form; for instance, a machine in which a large weight is hoisted by the application of a small effort may give rise to the general observation that the large weight moves through a smaller distance than does the point of application of the effort. It is not likely that this will be enunciated as a principle involving quantitative measure of work. Indeed the course as a whole may only be storing experiences from which principles such as the conservation of energy or momentum can later gain meaning.

Certainly the method of working from the general principle to the special case, which is the method of most mechanics textbooks, will not be generally applicable in the Modern School.

A valuable rule in approaching mechanics is to frame every question in terms of a familiar object, for instance, not a 'rotating body', but this 'rotating bicycle wheel'; not a 'lever', but the 'pedal shaft', or 'the beam of this balance', and so on. A particular pulley may be studied by neglecting or making allowance for friction, but it is not a 'frictionless (and imaginary) pulley'.

**7·401.** The following brief notes indicate some of the topics which might be considered when compiling a syllabus of work in mechanics. They would not be studied under the headings given here; some practical examples may very well illustrate two or more topics of widely differing natures.

### 7·41. Measurement.

Mechanics offers considerable scope for using or discussing units of measurement, degrees of accuracy, etc., while the measuring instruments are often based on mathematical notions of interest (e.g. vernier scale, micrometer screw gauge, steelyard). Many quantities are better measured in metric units; some need compound units such as speeds in feet per second, rotation in revolutions per minute or degrees per second.

### 7·42. Motion.

A study of motion might include the following topics:

(*a*) Ideas of uniform motion in a straight line; of uniform speed in paths other than a straight line, of changing speed and of changing velocity (as not being the same thing); rate of changing speed, and how it can be measured in feet per second per second, or other units; acceleration and retardation as rate of changing velocity and therefore defined in those units attached to a direction.

(*b*) Space-time and speed-time graphs: how they show velocity and acceleration. Velocity and acceleration as vectors; the triangle of velocities.

(*c*) Relative speeds: speeds of passing, of approach, of meeting in the same path or parallel paths. Relative velocities.

(*d*) Bodies having two independent motions, such as a passenger crossing the deck of a moving vessel, or an aircraft in a crosswind (7·25). The path of a projectile.

### 7·43. Forces

Some consideration of force and its measurement is not too difficult for able modern school pupils. Many everyday instances of equilibrium can be found, e.g. window supported through the sash cord by a weight. The principle of moments is an example of a notion which is in very common use (e.g. nutcrackers, pincers) yet which is not so often recognised in a general form. Centres of gravity, Hooke's law, work, power, etc., are other topics which can be dealt with in an elementary and experimental manner.

### 7·44. Machines and Mechanisms.

In the study of these will be found examples of many of the foregoing topics, together with others particular to machines. They are better studied as actual working examples rather than laboratory specimens. The following points of interest arise:

(*a*) Change of direction of applied force, e.g., pulley, lever.
(*b*) Comparison between load and effort.
(*c*) Comparison between distances moved by load and by effort.
(*d*) Effects of friction; reducing friction.
(*e*) The various types of simple machines, e.g. pulleys, lever, wheel and axle, inclined plane, screw, etc.

## 7·5. Statistics.

There have been numerous references in this report to the opportunities offered by a statistical approach to data collected in the classroom and in field work; the following are some of the statistical topics which have been mentioned. In 2·131 a technique for finding the average height of a group is described in detail, and in 3·221 a diagrammatic representation of data collected on the sea shore is given as an illustration of what should be possible in an exercise on field work. In 6·217 pictorial representation of large numbers and conclusions to be drawn from such diagrams are illustrated. In 6·82 accuracy in measurement is discussed, and some types of graphical representation of data are referred to in 6·46.

### 7·51. The Mean and the Scatter of a set of Data.

It is not unlikely that in considering the above or similar topics pupils may have come to realise that the common average by no means indicates all the properties of a set of data. For example, the two sets of numbers 8, 9, 10, 11, 12 and 2, 5, 7, 7, 29 both have the average 10, but they are evidently dissimilar, and the problem arises how best to describe *and measure* the dissimilarity. For the statistician the measurement of the 'scatter' of a set of observations is often more important than calculation of the average; when linked together the two provide valuable information about the whole set. There are many occasions when the subject of scatter can arise in the classroom, particularly with regard to the marks obtained in tests and examinations.

**7·511. Range.** When asked what they would suggest as a measure of scatter pupils almost always say the *range*, i.e. the difference between the greatest and smallest observations. Since the range is based on only two observations, and does not take into account the bunching or thinning of items in different parts of the range, it will be seen that it does not offer a very good measure of scatter. This provides a useful lead to the discussion of the *inter-quartile range*.

**7·512. Inter-quartile range.** If the whole of the observations are set out in ascending (or descending) order of size or value, the middle one is called the *median*, marking the division between upper and lower halves. If the two halves are sub-divided into two equal parts again, the point of division between the lowest and second quarters is called the *lower quartile*, and that between the third and highest quarters the *upper quartile*. The range between the lower and upper quartiles is known as the *inter-quartile range;* it includes the middle half of the observations and excludes the top and bottom quarters. Apart from the doubtful advantage of having split the observations into two equal halves and selecting the middle half as more typical of the whole (an assumption of doubtful validity), the inter-quartile range is not very valuable, since the quartiles are based only on particular points in the whole range of values.

**7·513. Standard deviation.** It is useful to consider observations in terms of their difference or *deviation* from the mean (or average) of the whole set. Observations greater than the mean have positive deviation and those below the mean, negative (6·152, 6·29). When attention has been focussed on the importance of including *all* the data, it is not improbable that difficulty may arise from the suggestion that the mean of the deviations would be a useful index; this is not easy to refute, the reason for its non-adoption lying in realms beyond the scope of school work. Some slight semblance of justification may be derived from the inconsistency of assigning signs + or − to deviations and then, if mean deviation is to be calculated, abandoning them. *Standard deviation* has the merit of using the values of all the observations without raising any difficulty in dealing with deviations with unlike signs. (It also has merits which cannot be appreciated at the school stage.) In any statistical work in school the pupil will be dealing with a small set of measurements (known to the statistician as a *sample*); for such a set the standard deviation is the '*root mean square*' of the deviations (Appendix 2). The calculation of a root mean square deviation may appear cumbersome at first sight, but once the principle is understood calculation may be eased by certain short cuts, though not many modern school children are likely to reach such a stage. Although the pupil will not be concerned, the teacher should be reminded that while the correct calculation for scatter of a limited set of values is the *sample* standard deviation, or root mean square, the estimation of a *population* standard deviation is slightly different. In view of its importance and also its unfamiliarity to some teachers in Modern Schools, the calculation of the standard deviation is set out in full in Appendix 2.

### 7·52. Data from Geometry.

Geometrical drawing and measurement offer many opportunities of getting sample distributions of observations, and keenness and improve, ment in both drawing and accuracy of measurement have been aroused in a class by such calculations. In fact, a class has been known to suggest that the standard deviation should be used in assessing marks for accuracy of drawing (Appendix 2).

### 7·53. Other Statistical Ideas.

The introduction of statistical ideas in the modern school course will be novel to many teachers, but the increasing importance of statistics in everyday life compels some attention. The aspect of statistics treated in the preceding paragraphs seems the one most likely to arise from the experimental mathematics now proving so fruitful in the development of mathematical appreciation in ordinary boys and girls, but there are other possibilities, and much exploratory work lies ahead. For example, many problems of *choice and chance* or *probability* are susceptible of treatment from first principles and demand no mathematical techniques beyond arithmetical multiplication and division. Sampling as a topic for discussion is mentioned in 3·221.

## CHAPTER 8

# THE HISTORY OF MATHEMATICS

### 8·0. The Relevance of History to Teaching.

The story of the development of Mathematics summarises and illustrates so much of what we have tried to say that we use it to provide us with a conclusion to this report.

In the Association's *Mathematics in Primary Schools* the following phrases occur:

'The language of numbers and measurement and shape . . . has grown through centuries.'

'Mathematical ideas are an inherited accumulation through many centuries. . . .'

'Mathematics is the tool and language of science. . . . It is important, therefore that the children should learn the part Mathematics has played in these achievements. . . .'

'The child's right to enter into the accumulated knowledge of mankind.'

These extracts give a clear suggestion that the child should be led to appreciate Mathematics as a living subject that began inspired by man's simple needs, and that has developed both as man has adventured in ideas and also as he has felt the increased demands of a modern civilisation. It is still available for the necessary everyday uses of the ordinary citizen as well as having wider cultural and technical values.

This being so, there must be much of value to both teacher and pupil in the history of the subject. There is, however, no place in the Modern School for a new topic, 'The History of Mathematics', as an isolated part of the syllabus. The whole field is very extensive but a selection of the easier parts of the story will provide much material that can be used profitably in the classroom when the occasion arises.

### 8·1. When and How History can be Used.

The story of Mathematics can serve at least three purposes.

Firstly, much of the information can be given direct to young people to stimulate their interest, the interest that is aroused by any story of discovery and adventure (4·2*b*).

Secondly, the story of the development of our units can be used to make many of their names and relationships seem logical, sensible and natural (3·212, 4·62).

Thirdly, the methods of the discoverers can suggest ways in which the pupil can rediscover the accepted truths.

Often need gave rise to experiment and experiment produced a working rule for use on other occasions. The sequence of problem, experiment, discovery and use is as valid and necessary for the pupil as it was for the original mathematicians, though the teacher sees to it that suitable problems occur at proper intervals.

Sometimes however sheer joy in the Mathematics itself and in the mental exercise of solving problems led to discovery and elaboration, only later found to have use and significance for any but the mathematicians themselves. This pattern of extending knowledge may also be followed on occasion by the learner, but in the Modern School more sparingly than with the really able pupil who delights in ideas for their own sake.

### 8·2. Where to find Material.

There are standard books on the history of Mathematics (8·8) from which a selection of material for classroom use can be made. Here we only refer to a few typical examples which can be used in the ways suggested above.

The growth of mathematical language and the importance of drawing attention to the derivation of many mathematical words have been mentioned in 6·1 where specific cases are instanced. The words themselves may hold a clue to the story behind them.

### 8·3. Some Examples of Suitable Material.

Our sign, +, for 'plus', is seen to be a simple development from the roughly scrawled mark indicating a *surplus* of weight in a consignment of merchandise.

What could be simpler than our sign for equality only just 400 years old? We can quote the words of its inventor, Robert Recorde, who said that nothing could be more equal than the two small parallel lines, =, he proposed to use.

Another little story, for the most part well known, is that concerning the origin of our familiar £ s. d. The £ came from the word *Libra* meaning *pound*, after *pound* had developed from *pondus* meaning *weight*. This connection between weight and money is also seen in the *shekel*, both a weight and a coin. From *solidus*, the Roman coin, came our *s.* for *shilling*. The old long form of the s, ʃ, was modified eventually into the straight line, /, as in 4/6d. for 4s. 6d., and to the compositor this type of bar, for money or for fractions, is called a *solidus*. Thus here the wheel has turned full circle.

Among other names and words the derivation of the following are interesting: digit, cipher, counter, bench, bank, cheque, calculate, pecuniary, acre, furlong, rod and rood.

Topics which have a significant history include: base of ten; the decimal point; Roman numerals; the symbol for zero; the calendar; 360 degrees; the measures of time, the minute part of an hour, the second minute part; Napier's rods; slide rule, logarithms.

### 8·4. The Growth of Algebra.

Again considering the history of Mathematics as giving a direct pointer to the child's method of learning, there are clear indications in the development of algebra.

This was originally purely rhetorical, and, without any symbolism, formulae were evaluated and equations solved with perfect understanding. As a single example of this, in solving what was a very popular type of problem with them, the ancient Egyptians used what we consider an algebraic method, by using the spoken word and no symbols at all, to find 'A quantity whose seventh part is added to it to become 19 '. The same rhetorical methods were used to solve problems concerning innumerable pipes flowing into a cistern, and also, surprisingly enough, problems which lead to quadratic equations. *After* this came the symbols. Those of Diophantus in the third century A.D. were a shorthand (usually termed syncopated algebra) which to us seems cumbersome, but it replaced words almost as if by a code. It was not until the time of Descartes, 1300 years later, that a symbolism similar to that of today had evolved, using $x$ for the unknown. The child's learning today can follow the same path. From complete verbal understanding and statement, he can proceed, via a code he has helped to invent, in which perhaps he chooses $n$ for a number but then takes $s$ shillings, $b$ boys, $l$ yards for length, $h$ feet for height, and so on until he can use $x$ as the unknown number whatever the units, and write down the data of a problem in the most abbreviated form.

### 8·5. The Growth of Geometry.

Similarly we know that Geometry began as a tool to do what the name geo-metry implies, that is, to measure land, and it began numerically from the analysis and dissection of the areas to be measured. From these working rules, discovered empirically from experiment, there developed the generalisations into formulae for the mensuration of various shapes. The properties of symmetrical shapes, and the synthesis of new shapes from known ones, added to the list of areas that could be calculated. None of this was deductive geometry, which was beginning elsewhere, at the same time, as a purely logical and intellectual exercise. Thus for the young learner, too, there will be the problem to be solved, the specific instances each attacked experimentally, the appearance of a general rule, and then, for those who need further intellectual stimulation and satisfaction, the deductive processes.

Among the many topics that lend themselves to this treatment the following might be mentioned here:

Thales; ratio, similar triangles, trigonometry; Pythagoras' Theorem as a pattern of shapes, as a numerical method of constructing a right angle, as a fact relating areas, as a generalised theorem; $\pi$ and the circle; coordinates and maps; square root as a problem in area; figurate numbers and series and properties of numbers; limits.

### 8·6. The Development of Measures.

The idea of gradual development has been stressed; similarly pupils may come to appreciate, for example, that the measures we use have grown from early man's sensible use of what was available—and for measures of length that was the body. There may then be point in contrasting this with the two measures which were deliberately man-made, the metre and Gunter's chain. These were made at a specific time for a special reason and purpose and owe nothing to tradition or to a gradual unfolding of methods of increasing accuracy.

### 8·7. Summary.

Finally, it is suggested that respect for the development of the subject can prove to be a touchstone of good teaching as well as a help to the development of the learner. We have tried throughout to keep this in mind and to remember that the subject is still developing and meeting new needs in a changing world.

### 8·8. Some Useful Books.

From the vast literature on the history of Mathematics, the following short list gives some which will be found interesting and informative:

(*a*) Particularly suitable as reference books for the pupils.
  * *Number Stories of Long Ago*, D. E. Smith, Ginn.
  * *Romance in Arithmetic*, M. E. Bowman, U.L.P.
  *Man Must Measure*, L. Hogben, Rathbone.
  *Stories of Mathematics*, S. E. Williams, Evans.

(*b*) Reference books for the teacher.
  *Man and Number*, D. Smeltzer, Black.
  *Mathematics for the Million*, L. Hogben, Allen & Unwin.
  * *Number the Language of Science*, T. Dantzig, Allen & Unwin.
  *A Short History of Mathematics*, W. W. R. Ball, Macmillan.
  * *A Short History of Mathematics*, V. Sanford, Harrap.
  * *A History of Elementary Mathematics*, F. Cajori, Macmillan.
  * *History of Mathematics*, Vols. I & II, D. E. Smith, Ginn.

The books marked * are included in the Association's *List of Books Suitable for School Libraries* mentioned in 5·72.

# APPENDIX 1

## DRAWING CERTAIN CURVES

*(a) Conics drawn as loci and as envelopes.*

(i) *The ellipse*

This curve can be traced by a pencil point, $P$, constrained by a loop of string held taut round two pins, $A$ and $B$ (the foci). This mechanical device is equivalent to the geometrical condition that the sum of $AP$ and $PB$ is constant, since the length of the loop and the length $AB$ are unchanged.

Ellipses of different shapes can be produced by varying the length $AB$ for a given loop. This series of ellipses includes the special case of the circle when $A$ and $B$ coincide and the radius is half the length of the loop.

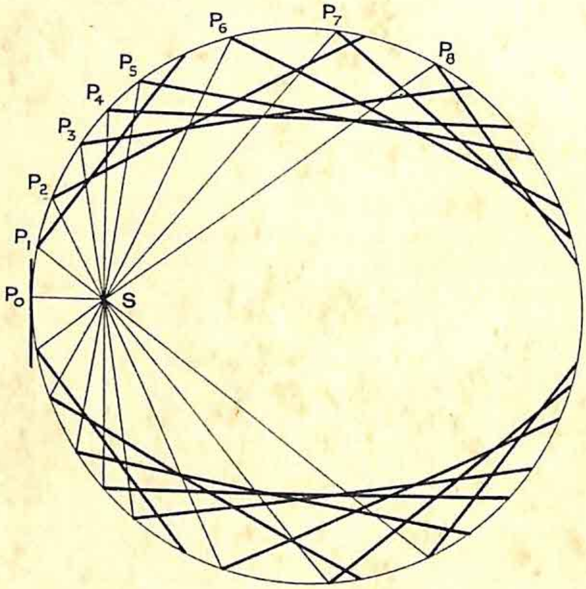


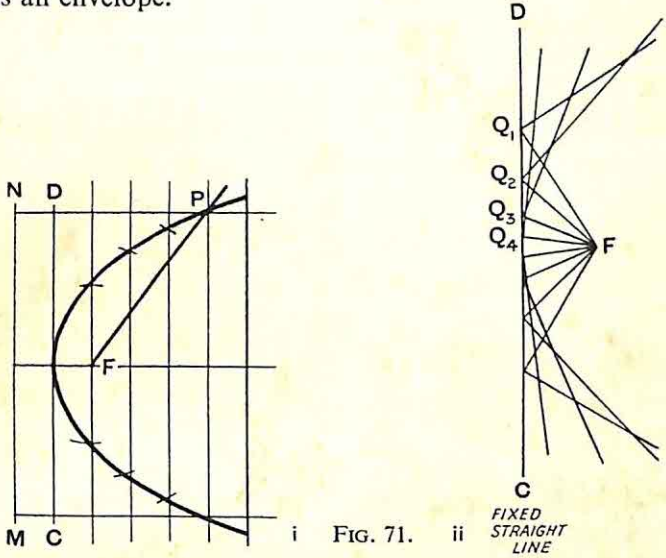
FIG. 70.

Figure 70 shows the ellipse drawn as an envelope as follows:

Draw a circle and take a point $S$, not the centre, within it. Join $S$ to points $P_0P_1P_2$, etc. on the circumference. Draw lines at right angles to $SP_0$, $SP_1$, $SP_2$, etc. These perpendiculars are tangents to the ellipse. ($S$ is a focus of the ellipse and the circle is called its auxiliary circle.) For convenience in drawing $S$ should be taken well away from the centre of the circle to obtain an ellipse with 'axes' of markedly different length.

(ii) *The parabola.*

Figure 71 shows methods of constructing the parabola (i) as a locus and (ii) as an envelope.



i FIG. 71. ii

$P$ is equidistant from $F$ (the focus) and a line $MN$.

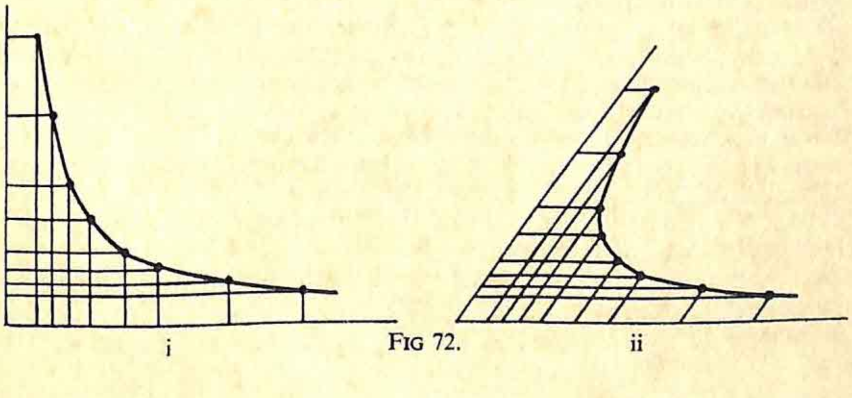
$P$ describes a parabola.

$Q$ moves along a fixed line $CD$; the perpendicular to $QF$ at $Q$ touches a parabola and so, as $Q$ moves, defines this parabola as an envelope.

(If $CD$ were drawn in Fig. 71 (i), it would bisect the perpendicular from $F$ to $MN$ and be parallel to $MN$.)

(iii) *The hyperbola.*

A good idea of the shape of a rectangular hyperbola can be obtained by fitting a large number of rectangles of the same area but different shape into one corner of a rectangular table. This is equivalent to plotting a graph of $xy = k$ (where $k$ is the area of the rectangles). The axes of this graph are called asymptotes of the hyperbola. The hyperbola has two branches. Figure 72 (i) shows one branch only of a rectangular hyperbola.



i FIG 72. ii

If the asymptotes are not at right angles, as in Fig. 72 (ii), the rectangles must be replaced by parallelograms fitting into the acute or obtuse angle between the lines.

A branch of the hyperbola as an envelope can be made by calculating corresponding points in which tangents cut the asymptotes, i.e. the lines which the branches of the hyperbola approach but never reach. For example, in Fig. 73 $OA \times OB = 72$ units. Pairs of values, say $2 \times 36$, $3 \times 24$, $4 \times 18$ etc. give positions of $AB$ and define the envelope.

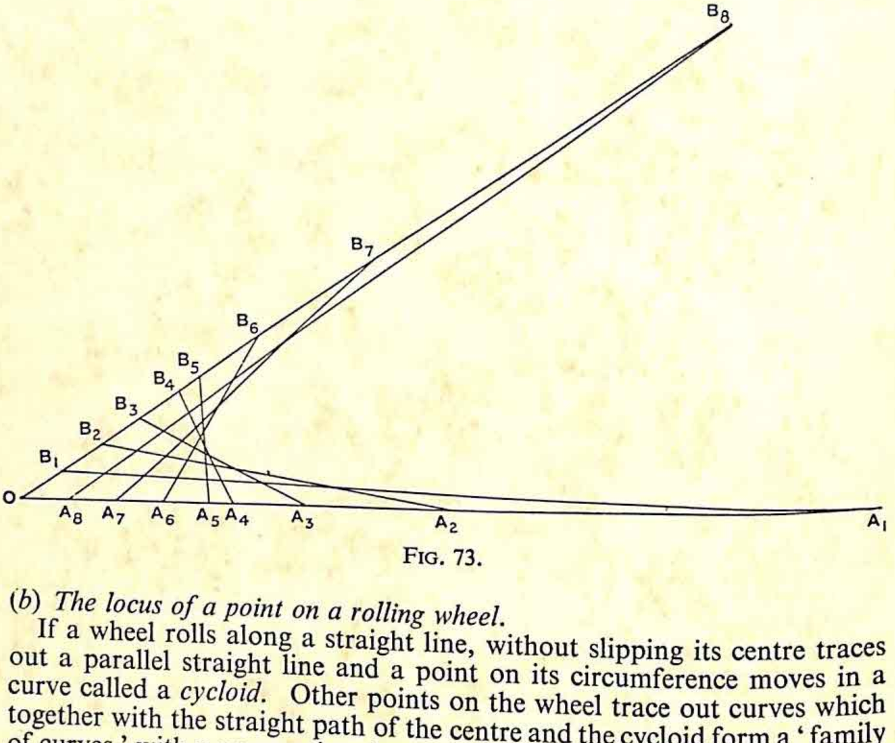


FIG. 73.

(*b*) *The locus of a point on a rolling wheel.*

If a wheel rolls along a straight line, without slipping its centre traces out a parallel straight line and a point on its circumference moves in a curve called a *cycloid*. Other points on the wheel trace out curves which together with the straight path of the centre and the cycloid form a ' family of curves ' with a progressive change of shape as the distance of the point from the centre increases.

Observation of a rolling wheel may give clues to the shapes of these curves, or a model may be devised, for example using a cardboard disc with a radius cut out as a slot, in which the position of a moving point can be recorded as the motion proceeds (6·14).

The cycloid can be plotted geometrically as follows:

Suppose the moving point $P$ on the circumference is initially in contact at $O$ with a line $OA$ along which the circle rolls.

Mark off a segment $OA$ equal to the circumference of the circle.

Divide $OA$ into equal segments, say eight, at $A_1, A_2, A_3 \ldots$

Draw the circle representing the wheel in contact with $OA$ at each of these points $A_1, A_2, A_3 \ldots$

Suppose $P$ to have moved to $P_1, P_2, P_3 \ldots$ on these circles.

Then $P_1$, $P_2$, $P_3$, ... are points on the cycloid which can be found by dividing the angle at the centre of the circle into eight parts and marking

$P_1$ on the circle at $A_1$ at the first eighth,
$P_2$ on the circle at $A_2$ at the second eighth,
..........................................and so on.

A model may be needed to show how $P$ moves to its earlier positions, when although $P$ has moved forward it lies behind the centre of the circle.

(*c*) *The locus of a point $P$ whose distance from a fixed point $O$ increases by equal amounts for equal rotations of $OP$.*

This curve is a spiral. A half turn of a curve of this type (and its reflection) would be suitable as an outline of a cam, such as that in a sewing machine winder, to control a regular to and fro motion associated with regular rotation.

The curve can be constructed as follows:

A revolving radius (radius vector) drawn from a pointa $O$increases in length by equal amounts, say $x$ units, for equal increases of angle.

Draw a number of lines radiating from $O$ at equal angles.

If a beginning is made when the radius is 0, mark off points distant $x$, $2x$, $3x$ ... units from $O$ on the successive radiating lines.

The lines may be drawn at first at right angles through $O$. If $x$ is the increment corresponding to a turn of a right angle, intermediate points may be obtained by using fractions of $x$ for fractions of a right angle.

The equation of the curve is $r = a\theta$, where $r$ is the length of the revolving radius and $\theta$ the angle through which it has turned, (in the standard equation $\theta$ is measured in radians).

## APPENDIX 2

# CALCULATION OF STANDARD DEVIATION

If $p, q, r$ ... form a set of $n$ values, the standard deviation $\sigma$ (sigma) is given by

$$\sigma=\sqrt{\frac{d_p^2+d_q^2+d_r^2+\dots}{n}}$$

where $d_p, d_q, \dots$ are the deviations of $p, q, \dots$ from the mean of the set.

As examples of the principle, the standard deviations of the two sets given in 7·51 would be:

*Set I*

8, 9, 10, 11, 12, the mean being 10;
$d$: $-2, -1, 0, 1, 2$
$d^2$: $4, 1, 0, 1, 4$

$$\sigma=\sqrt{\frac{4+1+0+1+4}{5}}=\sqrt{2}=1{\cdot}414.$$

*Set II*

2, 5, 7, 7, 29, the mean again being 10;
$d$: $-8, -5, -3, -3, 19$
$d^2$: $64, 25, 9, 9, 361$

$$\sigma=\sqrt{\frac{64+25+9+9+361}{5}}=9{\cdot}674.$$

(It should be noted that these examples are given only to illustrate the basic computation; they have no practical significance.)

It will be seen from the above examples that the basic steps are the computation of

(i) mean of set;
(ii) deviations of separate items from mean;
(iii) squares of separate deviations;
(iv) sum of squares of deviations;
(v) division by number of items;
(vi) square root of (v).

As the mean of an actual set is not usually a whole number, the work of finding deviations, and particularly of squaring them, would be very laborious if the calculation were worked from first principles. In practice, calculation is shortened by *assuming* an arbitrary mean (usually close to the expected value of the true mean, though this is not necessary); the deviations are then more easily manipulated, and a correction applied to refer the deviations to the true instead of the arbitrary mean. Another short cut is achieved by grouping the observations in short equal intervals over the whole range of values; the values in each group are then taken as if they were coincident at the mid-point of the interval. Handling the work is also helped by setting it out in tabular form. The procedure is illustrated in detail in the following section, the example being taken from the actual work of a class.

Measurement by 26 boys of the lengths of the six chords of a regular hexagon of side 2 inches (to nearest hundredth of an inch)

| Length (inches) | Frequency ($f$) | Arbitrary scale of Deviations ($x$) | ($fx$) | ($fx^2$) |
|---|---|---|---|---|
| 3·54–3·52 | 17 | 2 | 34 | 68 |
| 3·51–3·49 | 43 | 1 | 43 | 43 |
| 3·48–3·46 | 61 | 0 | 0 | 0 |
| 3·45–3·43 | 30 | $-1$ | $-30$ | 30 |
| 3·42–3·40 | 5 | $-2$ | $-10$ | 20 |
| | 156 | | 37 | 161 |

Col. 1 indicates that the 156 measurements ranged from 3·40 to 3·54 inches. This range is then divided into 5 intervals 3·40–3·42, 3·43–3·45, etc. A suitable number of intervals is usually 5 to 9.

Col. 2 gives the number of measurements falling in each interval. The total is entered as a check on the counting of the items in each group.

In col. 3 the arbitrary mean has been chosen as the mid-point of the 3·46–3·48 interval: 3·47, but further calculation is simplified by taking this as the origin of a new arbitrary scale of deviations, the scale units being whole intervals (each of 0·03 inch). The rest of the work in the table is in terms of the arbitrary mean and deviation units and is therefore entirely in whole numbers. The rest of the tabulated work is directed towards finding the true mean (from col. 4) and the sum of squares of deviations (from col. 5).

The figures in col. 4 are the products of those in cols. 2 and 3. The total of col. 4 is the total of deviations from the arbitrary mean: if the arbitrary mean had been the true mean, the total of col. 4 would have been zero. As there are 156 items, the average deviation (from the arbitrary mean) is

$$\frac{+37}{156}\ \textit{scale intervals}, \text{ i.e. } \frac{37}{156}\times 0{\cdot}03 \text{ inch}, \ = 0{\cdot}0071 \text{ inch}.$$

$$\therefore \text{ true mean } = (3{\cdot}47 + 0{\cdot}0071) \text{ in.} = 3{\cdot}4771 \text{ in.}$$

If to those who are unfamiliar with it this seems a roundabout way of calculating an average, it may be pointed out that it is in principle the method used when finding the average of several numbers between, say, 100 and 120, when one takes out the common part, 100, and finds the average of the residues; in this case the arbitrary mean is 100, and the average of the residues is the average deviation from the arbitrary mean. In the present work the procedure has the important advantage that it leads on easily to the calculation of the standard deviation $\sigma$.

Col. 5 is the product of cols. 3 and 4, i.e. $fx^2$ is reached as $fx \times x$ instead of, more obviously, as $f \times x^2$. The total of col. 5, 161, is the sum of the squares of the deviations from the arbitrary mean, and to reduce this to the least sum of the squares of the deviations from the true mean a correction has to be *subtracted*,* equal to $\frac{37^2}{156}$, i.e. the square of the total deviations (col. 4) divided by the number of items.

$$\begin{aligned}
\text{Corrected sum of squares} &= 161 - \frac{(37)^2}{156} \\
&= 161 - 8{\cdot}776 \\
&= 152{\cdot}224. \\
\text{Mean square of deviations} &= \frac{152{\cdot}224}{156} \\
&= 0{\cdot}9758.
\end{aligned}$$

(This is known as the *Variance*.)

$$\begin{aligned}
\text{Standard deviation } \sigma &= \sqrt{0{\cdot}9758} \\
&= 0{\cdot}9878.
\end{aligned}$$

But this value is in terms of scale intervals of 0·03 in.

$$\begin{aligned}
\therefore\ \sigma &= (0{\cdot}9878 \times 0{\cdot}03) \text{ in.} \\
&= 0{\cdot}0296 \text{ in.}
\end{aligned}$$

The set of 156 observations is therefore described as having a mean of 3·4771 in. with a standard deviation of 0·0296 in. It is interesting to note that the correct value should have been $\sqrt{12}$ in., or 3·4641 in.

---

* The reason why this correction always involves a *subtraction* may not be apparent, but the statement may be checked by re-working with a different arbitrary scale of deviations:

(i) if the scale of col. 3 is ' moved down ' one unit, to range from $-1$ to 3 instead of $-2$ to 2, fx and $fx^2$ columns totals become 193 and 391 respectively, and the corrected least sum of squares $= 391 - \frac{(193)^2}{156} = 152{\cdot}224$

(ii) if the scale is ' moved up ' one unit, the corrected least sum of squares becomes

$$243 - \frac{(-119)^2}{156},\ = 152{\cdot}224.$$

The teacher may refer to one of the standard books on statistics, such as Chambers: Statistical Calculations.

# INDEX

## OF TERMS AND TOPICS IN PART II WITH REFERENCES TO PART I

The reference numbers are to *paragraphs;* the leading figure is that of the chapter, but decimal points have been omitted. *Principal references* are in heavy type. Where an argument continues over *consecutive paragraphs* only the first is named. Terms, such as rectangle, which occur frequently are not included, nor are many minor references.

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